## Chapter VI. Fluid description of edge plasma transport

Tokamak edge plasma is especially difficult for modeling – it is the interface between the hot core plasma and relatively low-temperature divertor region where multiple impurity species and neutral gas dynamics are important. The presence of material surfaces, e.g. divertor plates, with additional surface interactions and reactions, complex magnetic field geometry, such as separatrix, and strong electric field and plasma flows make it even more complex. Additional complications are introduced by the presence of non-neutral regions (boundary sheath) near the plasma-wall boundaries. In general, the plasma edge is the region where multiple collision and atomic processes having very different characteristic times and lengths are equally important and need to be considered self-consistently and simultaneously with the anomalous turbulent transport phenomena. Clear separation of the time and length scales, e.g. between the equilibrium and fluctuating quantities, is often impossible in the edge region, which creates additional challenges.

The plasma turbulence and anomalous transport remain the biggest challenge for the physics of edge plasmas. Currently, it is not feasible to simulate plasma turbulence in the edge and fully include all possible kinetic effects, neutrals and atomic physics, sheath, boundaries, etc. Such simulations are still outside the modern computer capabilities. Therefore, a number of simplifications and reductions of the problem are usually performed. Presently, one can identify two major directions in the theoretical description and modeling of the plasma edge. In one approach, which is conditionally called here the first principle turbulence modeling, the focus is on formulating the adequate physics models, which would include relevant physics at small scales to describe properly instabilities, turbulence and anomalous transport (see Ch. VII for further discussions). The global turbulence codes are being developed, which aim to characterize nonlinear plasma fluctuations and transport at the edge of the magnetic confinement devices, including 3D effects and open magnetic field geometries [1], [2], [3], [4], [5], [6], [7], [8], [9]. Because of limitations noted above, such codes are often based on fluid (moment) formulation and neglect interactions with neutrals and many kinetic effects such as parallel transport.

In the alternative approach (e.g. see [10], [11], and Ch. VIII), the emphasis is on the characterization of large spatiotemporal scale equilibria and flows of particles and energy in complex divertor geometries including coupling to neutrals, sheath boundaries, atomic physics, plasma surface interactions, etc. Such codes, conditionally called here transport codes, include the effects of small-scale fluctuations and anomalous transport by using mostly empirical anomalous transport coefficients. The exact structure of anomalous transport, i.e. the form of the transport matrix and thermodynamical forces responsible for anomalous transport, e. g. the pinch effects, the role and the form of the residual stress, etc. is the subject of intense studies and debates. The most common approach is to add to the classical coefficients for perpendicular transport some empirical anomalous values (see [10], [11], [12] and Ch. VIII).

In this chapter, we highlight the description of plasma dynamics based on the fluid (moment) approach, the main assumptions, the validity limits, and the discussion of how the moment approach is used in the transport codes. The strong magnetic field of fusion devices allows certain classification of the cross-field particle, momentum and energy fluxes, as well as some simplifications of the resulting equations governing these quantities. In particular, the collisionless cross-field fluxes play an important role in defining the electric field, the parallel current and the flows in edge plasmas and are currently included in the transport codes used to model the plasma edge [10], [11], [12]. We discuss also plasma transport driven by the inhomogeneity of the plasma parameters along the magnetic field, which are of particular importance in the SOL region where the magnetic field lines intersect material surfaces.

### VI.1 Hierarchy and closure of the fluid equations

We introduce here the basic moment (fluid) equations to fix the notations and define the relevant variables. In the most general form, the dynamics of electrons, ions, and neutral species can be described with the kinetic Boltzmann equations

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\nabla_{\vec{v}} f_\alpha = \sum_\beta C(f_\alpha, f_\beta), \tag{VI.1}$$

for the distribution functions $f_\alpha(\vec{v},\vec{r},t)$ of all species $\alpha$ (characterized by the charge $e_\alpha$ and mass $m_\alpha$), including the neutral particles. This would be the most comprehensive approach for plasma modeling. The collision integrals $C(f_\alpha, f_\beta)$ on the right-hand side of Eq. (VI.1) must include all inter-particle and self-collisions as well as various atomic processes such as ionization, charge-exchange, and others. The electric, $\vec{E}$, and magnetic, $\vec{B}$, fields in Eq. (VI.1) need to be determined self-consistently from the Maxwell equations with the electric charges and current sources found from the solution of the kinetic equations (VI.1). Solving all these equations can only be possible numerically. However, in the near future, this is not feasible without a significant reduction of these equations. In general, the fluid equations themselves are the examples of such a reduction when the time evolution of the six-dimensional distribution function is replaced by a truncated set of nonlinear equations for space and time evolution of the moments, $\hat{M}(\vec{r},t)$, of the distribution function $F(\vec{v},\vec{r},t)$, defined as the integrals $\hat{M}(\vec{r},t) = \int \hat{\hat{M}}(\vec{v})F(\vec{v},\vec{r},t)d\vec{v}$. We notice that in general case, both $\hat{\hat{M}}(\vec{v})$ and $\hat{M}(\vec{r},t)$ can be tensors. The first moments of the distribution function $F(\vec{v},\vec{r},t)$ are the basic fluid variables such as the particle density, $n(\vec{r},t)$, the average (fluid) velocity, $\vec{V}(\vec{r},t)$, and the pressure, $p(\vec{r},t)$, that have simple macroscopic meaning:

$$n(\vec{r},t) = \int F(\vec{v},\vec{r},t)d\vec{v}, \tag{VI.2}$$

$$n(\vec{r},t)\vec{V}(\vec{r},t) = \int \vec{v}F(\vec{v},\vec{r},t)d\vec{v}, \tag{VI.3}$$

$$p(\vec{r},t) = \frac{m}{3}\int v'^2 F(\vec{v}',\vec{r},t)d\vec{v}, \tag{VI.4}$$

where $\vec{v}' = \vec{v} - \vec{V}(\vec{r},t)$. As one can see from Eq. (VI.4), the pressure is expressed in terms of the "random" particle velocity $\vec{v}' = \vec{v} - \vec{V}(\vec{r},t)$, which corresponds to the particle velocity in the reference frame of the fluid velocity $\vec{V}(\vec{r},t)$. We will see that this random velocity will be used in other moments of the distribution function. Therefore, it is useful to re-write the kinetic equation (VI.1) by using the variable $\vec{v}'$ instead of $\vec{v}$. After some algebra we find

$$\frac{df_\alpha}{dt} + \vec{v}'\cdot\nabla f_\alpha + \left\{\frac{e_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right) - \frac{d\vec{V}}{dt}\right\}\cdot\nabla_{\vec{v}'} f_\alpha - \frac{\partial V_i}{\partial x_k} v'_k \frac{\partial f_\alpha}{\partial v'_i} + \frac{e_\alpha}{m_\alpha c}\frac{\vec{v}'\times\vec{B}}{c}\cdot\nabla_{\vec{v}'} f_\alpha = \sum_\beta C(f_\alpha, f_\beta), \tag{VI.5}$$

where $f_\alpha = f_\alpha(\vec{v}',\vec{r},t)$, and $d(...)/dt = \partial(...)/\partial t + \vec{V}\cdot\nabla(...)$.

The evolution equations for the fluid variables are obtained in a standard way [13] by taking the moments of Eq. (VI.5) with appropriate weights, 1, $m\vec{v}'$, and $mv'^2/2$, which leads to the sequence of the equations

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{V}) = C_n, \tag{VI.6}$$

$$\frac{\partial(mn\vec{V})}{\partial t} + \nabla\cdot(mn\vec{V}\vec{V}) = -en\left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right) - \nabla p - \nabla\cdot\vec{\Pi} + \vec{C}_V, \tag{VI.7}$$

$$\frac{\partial}{\partial t}\frac{3}{2}p + \nabla\cdot\left(\frac{3}{2}p\vec{V} + \vec{q}\right) + p\nabla\cdot\vec{V} + \vec{\Pi}:\nabla\vec{V} = C_p, \tag{VI.8}$$

where the terms $C_n$, $\vec{C}_V$, and $C_p$ are the respective moments of the collision operators resulting in the sources and sinks of the particle density, momentum and energy, which need to be specified for each species, and

$$\vec{q}(\vec{r},t) = \int \vec{v}'\frac{m}{2}v'^2 f(\vec{v}',\vec{r},t)d\vec{v}', \tag{VI.9}$$

$$\vec{\Pi} = \int m\left(\vec{v}'\vec{v}' - \frac{v'^2}{3}\vec{I}\right)f(\vec{v}',\vec{r},t)d\vec{v}', \tag{VI.10}$$

where $\vec{I}$ is the identity tensor. As we see from Eqs. (VI.8), (VI.9), vector $\vec{q}(\vec{r},t)$ and tensor $\vec{\Pi}(\vec{r},t)$ describe, correspondingly, the particle energy and momentum fluxes in the moving frame. Hereafter we omit for simplicity the indices $\alpha, \beta, ...$ defining different species.

To "close" the system of equations (VI.5)-(VI.7), we need to express $\vec{q}$ and $\vec{\Pi}$ (as well as the moments of the collision operator $C_n$, $\vec{C}_V$, and $C_p$) in terms of the density, average velocity and pressure. Within the fluid description, this is only possible by assuming that the Coulomb collisions of charged particles of the same species are fast enough, so that the distribution function $f(\vec{v}',\vec{r},t)$ is close to the Maxwellian with the temperature $T$ (which can be different for the species with a large mass difference). Quantitatively, this feature should be related to the smallness of some parameter(-s). For the case with no magnetic field, such small parameters, $\varepsilon << 1$, are $\lambda_C / L$ and $\omega / \nu_C$, where $\nu_C$ is the frequency of the Coulomb collisions of species $\alpha$, $\lambda_C = \sqrt{T/m}/\nu_C$ is the mean free path between such collisions, $L$ is the spatial scale of the inhomogeneity of the plasma parameters and $\omega$ is the characteristic frequency of their temporal variation. However, the dynamics of plasma embedded in a strong magnetic field with $\Omega_B >> \nu_C$, where $\Omega_B = eB/mc$ is the cyclotron frequency, becomes very anisotropic. As a result, the small parameters allowing for the spatial inhomogeneity of magnetized plasma parameters within the fluid approximation become (e.g. see [13]) $\lambda_C / L_{||}$ and $\sqrt{\lambda_C \rho}/L_\perp$, where $\rho = \sqrt{T/m}/\Omega_B$ is the particle Larmor radius whereas $L_{||}$ and $L_\perp$ correspond to the plasma parameter inhomogeneity along and across the magnetic field (we assume here that $L_\perp$ defines the inhomogeneity of both the plasma parameters and the magnetic field). We notice that for $\rho / L_\perp << 1$, the cross-field plasma dynamics is reasonably well described by fluid type equations even for collisionless plasmas (e.g. see [14]).

In what follows we will assume that $\varepsilon << 1$ and $f(\vec{v}',\vec{r},t) \cong F_{Max}(n,T,\vec{v}',\vec{r},t)$. In this case, there are two major approaches to utilize this small parameter in the derivation of $\vec{q}$, $\vec{\Pi}$, and the moments of the collision operator $C_n$, $\vec{C}_V$, and $C_p$. Both of them are based on the representation of the distribution function in the moving frame as

$$f(\vec{v}',\vec{r},t) = F_{Max}(n,T,\vec{v}',\vec{r},t)\left(1+\Phi(\vec{v}',\vec{r},t)\right), \qquad \text{(VI.11)}$$

where $F_{Max}(n,T,\vec{v}',\vec{r},t)$ is the dynamic Maxwellian distribution function whereas $\Phi$ is small, $|\Phi| << 1$, and, in addition, does not contribute to the particle density, average velocity and pressure (temperature, $T = p/n$). In both approaches, the expression (VI.11) is substituted in Eq. (VI.5). However, the further steps in "closing" the equations (VI.6)-(VI.8) are different.

In the Chapman-Enskog approach [15] (adopted for "simple", one ion species plasma by Braginskii [13]), in zero-order approximation in small parameter $\varepsilon$, an impact of $\Phi$ on particle transport is completely ignored and the evolution of $n(\vec{r},t)$, $\vec{V}(\vec{r},t)$, and $p(\vec{r},t)$ is described by Eqs. (VI.6)-(VI.8) with no $\vec{q}$, $\vec{\Pi}$, and the moments of the collision operator. This allows expressing the zero-order time derivative of $n(\vec{r},t)$, $\vec{V}(\vec{r},t)$, and $T(\vec{r},t)$ in terms of their spatial derivatives. Then, in the first-order approximation, $\Phi$ is only retained in the largest terms describing gyro-rotation and collision operators (linearized over $\Phi$), whereas all other terms are expressed via the spatial derivatives of $n(\vec{r},t)$, $\vec{V}(\vec{r},t)$, and $T(\vec{r},t)$. Finally, this nonhomogeneous linear integrodifferential equation for $\Phi$ is solved by representing $\Phi$ in the series of tensorial expansion

$$\Phi = \Phi_0 + \Phi_i v'_i + \Phi_{ik}\left(v'_i v'_k - \frac{v'^2}{3}\delta_{ik}\right) + \dots, \qquad \text{(VI.12)}$$

whereas the coefficients of this expansion are written as infinite series

$$\Phi_0(\vec{r}, v'^2) = \sum_{j=2}^{\infty} a_0^{(j)}(\vec{r}) L_j^{(1/2)}\left(v'^2 / v_T^2\right), \qquad \text{(VI.13)}$$

$$\Phi_i(\vec{r}, v'^2) = \sum_{j=1}^{\infty} a_i^{(j)}(\vec{r}) L_j^{(3/2)}\left(v'^2 / v_T^2\right), \qquad \text{(VI.14)}$$

$$\Phi_{ik}(\vec{r}, v'^2) = \sum_{j=0}^{\infty} a_{ik}^{(j)}(\vec{r}) L_j^{(5/2)}\left(v'^2 / v_T^2\right), \qquad \text{(VI.15)}$$

where $L_j^{(\alpha)}(x)$ are the generalized Laguerre (or Sonin-Laguerre) polynomials and $v_T^2 = 2T/m$ (e.g. see [16], [13], [17] and the references therein). We notice that all the terms in the expansion (VI.12)-(VI.15) are orthogonal to each other (with the weight function proportional to $F_{Max}(n, T, \vec{v}', \vec{r}, t)$). Moreover, taking into account such orthogonality, we find that $\Phi$ does not contribute to $n(\vec{r}, t)$ and $T(\vec{r}, t)$ whereas implementing Eq. (VI.12) into Eq. (VI.9), (VI.10) it is easy to show that the components of the energy flux $q_i$ and tensor $\Pi_{ik}$ can be expressed through the vector $a_i^{(1)}$ and tensor $a_{ik}^{(0)}$:

$$q_i(\vec{r}, t) = \int v'_i \frac{m}{2} v'^2 L_1^{(3/2)}\left(v'^2 / v_T^2\right) f(\vec{v}', \vec{r}, t) d\vec{v}' = -\frac{2}{5}\frac{pT}{m} a_i^{(1)}, \qquad \text{(VI.16)}$$

and

$$\Pi_{ik} = \frac{pT}{m} a_{ik}^{(0)}. \qquad \text{(VI.17)}$$

However, the orthogonal functions used for the decomposition of $\Phi(\vec{v}', \vec{r}, t)$ in (VI.12)-(VI.15) are not eigenfunctions of the linearized collision operators. Therefore, the components of $\Phi_i(\vec{r}, v'^2)$ contribute not only to the heat flux $\vec{q}(\vec{r}, t)$ but also to $\vec{C}_V$ which results in the so-called thermal forces depending on $\nabla T$ (see [13] for the discussion of the physical nature of the thermal force). Thus, we arrive at the sets of an infinite number of equations for the coefficients $a_i^{(j)}$ and $a_{ik}^{(j)}$. The solutions of these equations can only be found by keeping only a finite number of these coefficients. But the accuracy of physically meaningful parameters $a_i^{(1)}$ and $a_{ik}^{(0)}$, determining the energy and momentum fluxes (VI.16), (VI.17) and contributing to $C_n$, $\vec{C}_V$

, and $C_p$ depends on the number of the terms kept in these sets of equations. In [13], [18] it was found that sufficient accuracy of $a_i^{(1)}$ and $a_{ik}^{(0)}$ can be reached by considering only two extra terms in Eq. (VI.14), (VI.15) proportional, respectively, to $a_i^{(2)}$ and $a_{ik}^{(1)}$, which define the following quantities

$$\vec{q}^*(\vec{r},t) = \int \vec{v}' \frac{m}{2} v'^2 L_2^{(3/2)}\left(v'^2 / v_T^2\right) f(\vec{v}',\vec{r},t) d\vec{v}', \tag{VI.18}$$

and

$$\vec{\vec{\Pi}}^*(\vec{r},t) = \int m\left(\vec{v}'\vec{v}' - \frac{v'^2}{3}\vec{\vec{I}}\right) L_1^{(5/2)}\left(v'^2 / v_T^2\right) f(\vec{v}',\vec{r},t) d\vec{v}'. \tag{VI.19}$$

We notice that in strongly magnetized plasmas, sufficient accuracy of collisionless components of $\vec{q}$ and $\vec{\vec{\Pi}}$, which are associated with the drift of particles caused by inhomogeneity of the magnetic field, can be reached by keeping only $a_i^{(1)}$ and $a_{ik}^{(0)}$.

Although the Chapman-Enskog approach works rather well for “simple” plasma, it becomes too cumbersome for multispecies plasma typical for edge plasmas. In addition, within the Chapman-Enskog approach, some additional effort is needed to recover the contribution of the heat flux $\vec{q}$ to the viscosity tensor $\vec{\vec{\Pi}}$, which is important for both edge plasma transport and turbulence studies [19], [20], [21].

An alternative to the Chapman-Enskog method is the Grad approach [22]. In the Grad approach, the component of the distribution function proportional to $\Phi$, recall Eq. (VI.11), is kept in both linearized collision operators and all other terms in Eq. (VI.5). Then from the equation (VI.5) one can obtain the hierarchy of the evolution equations for the higher-order moments by multiplying Eq. (VI.5) by $\vec{v}' m v'^2 / 2$, $m\left(\vec{v}'\vec{v}' - v'^2\vec{\vec{I}} / 3\right)$, etc. and integrating it over the velocity space. Such a set of the evolution equations can be truncated at some high moment, thus resulting in a closed set of the fluid equations. For example, we find the following evolution equations for the heat $\vec{q}$ and momentum $\vec{\vec{\Pi}}$ fluxes neglecting all higher moments:

$$\begin{aligned} &\frac{d\vec{q}}{dt} + \frac{7}{5}\vec{q}\nabla\cdot\vec{V} + \frac{7}{5}(\vec{q}\cdot\nabla)\vec{V} + \frac{2}{5}(\vec{q}\nabla)^{Tr}\vec{V} - \frac{e}{m}\vec{E}\cdot\vec{\vec{\Pi}} \\ &\qquad + \frac{7}{2m}\vec{\vec{\Pi}}\cdot\nabla T + \frac{T}{m}\nabla\cdot\vec{\vec{\Pi}} + \frac{5}{2}\frac{p}{m}\nabla T - \vec{q}\times\vec{\Omega}_B = C_{\vec{q}} \end{aligned}, \tag{VI.20}$$

$$\begin{aligned} &\frac{d\vec{\vec{\Pi}}}{dt} + \vec{\vec{\Pi}}\nabla\cdot\vec{V} + \vec{\vec{\Pi}}\cdot\nabla\vec{V} + \vec{\vec{\Pi}}\cdot(\nabla\vec{V})^{Tr} - \frac{2}{3}\vec{\vec{\Pi}}:(\nabla\vec{V}) - \left(\vec{\vec{\Pi}}\times\vec{\Omega}_B - \vec{\Omega}_B\times\vec{\vec{\Pi}}\right) \\ &\qquad = -p\left(\nabla\vec{V} + (\nabla\vec{V})^{Tr} - \frac{2}{3}\vec{\vec{I}}\nabla\cdot\vec{V}\right) - \frac{2}{5}\left(\nabla\vec{q} + (\nabla\vec{q})^{Tr} - \frac{2}{3}\vec{\vec{I}}\nabla\cdot\vec{q}\right) + C_{\vec{\vec{\Pi}}} \end{aligned}, \tag{VI.21}$$

where $\vec{\Omega}_B = \Omega_B \vec{B} / B$ whereas $C_{\vec{q}}$ and $C_{\vec{\Pi}}$ are the corresponding moments of the collision operators [22], [23], [17]. We note, however, that for the calculation of the collision-driven energy and momentum flux with sufficient accuracy, the evolution equation of the higher-order Laguerre polynomials, corresponding to $\vec{q}^*(\vec{r},t)$ and $\vec{\Pi}^*(\vec{r},t)$, should be considered [13].

### VI.2. Collisionless cross-field components of energy and momentum fluxes

For strongly magnetized plasmas, $\Omega_B >> \nu_C$, the leading terms in the fluid equations (VI.6)-(VI.8) and (VI.20), (VI.21) are those proportional to the magnetic field strength B. This allows solving the momentum balance equation (VI.7) for the cross-field component of $\vec{V}$ by expanding $\vec{V}_\perp$ in the powers of $1/B$ and neglecting the impact of the collision operators. Observing Eq. (VI.7), one finds that in the first order in $1/B$, there is a contribution from both the electric field and the pressure gradient:

$$\vec{V}_\perp^{(1)} = \vec{V}_E + \vec{V}_p, \tag{VI.22}$$

where

$$\vec{V}_E = c(\vec{E} \times \vec{B}) / B^2 \text{ and } \vec{V}_p = c(\vec{B} \times \nabla p) / enB^2 \tag{VI.23}$$

describe, respectively, the $\vec{E} \times \vec{B}$ and diamagnetic drift velocities. In the second order, we find the components of $\vec{V}_\perp$ related to inertial and collisionless viscosity polarization

$$\vec{V}_\perp^{(2)} = \vec{V}_I + \vec{V}_\Pi , \tag{VI.24}$$

where

$$\vec{V}_I = \frac{1}{\Omega_B} \frac{\vec{B}}{B} \times \left\{ \left[ \frac{\partial}{\partial t} + \left( \vec{V}_E + \vec{V}_p \right) \cdot \nabla \right] \left( \vec{V}_E + \vec{V}_p \right) \right\}, \tag{VI.25}$$

and

$$\vec{V}_\Pi = \frac{1}{\Omega_B} \frac{\vec{B}}{B} \times \nabla \cdot \vec{\Pi} . \tag{VI.26}$$

Similar expansion for $\vec{q}_\perp$ gives the following first-order expression.

$$\vec{q}_\perp^{(1)} = \frac{5}{2} \frac{cp}{eB^2} (\vec{B} \times \nabla T) . \tag{VI.27}$$

The first order term for collisionless (gyro-viscous) momentum flux $\vec{\Pi}$ can be found from the cross-field components of Eq. (VI.21), which gives the following equation

$$\begin{aligned} \hat{K}(\vec{\Pi}) &\equiv \vec{\Pi} \times \vec{\Omega}_B - \vec{\Omega}_B \times \vec{\Pi} \\ &= p \left\{ \nabla \vec{V} + \left( \nabla \vec{V} \right)^{Tr} - \frac{2}{3} \vec{I} \nabla \cdot \vec{V} \right\} + \left\{ \nabla \vec{q} + \left( \nabla \vec{q} \right)^{Tr} - \frac{2}{3} \vec{I} \nabla \cdot \vec{q} \right\} . \end{aligned} \tag{VI.28}$$

The inversion of the operator $\hat{K}(\vec{\Pi})$ gives the following expression for the gyro-viscous momentum flux

$$\vec{\Pi}_g = \frac{1}{4\Omega_B}\left\{\vec{b}\times\vec{W}\cdot(\vec{I}+3\vec{b}\vec{b})-(\vec{I}+3\vec{b}\vec{b})\cdot\vec{W}\times\vec{b}\right\}, \tag{VI.29}$$

where $\vec{b}=\vec{B}/B$ and

$$\vec{W} = p\left\{\nabla\vec{V}+(\nabla\vec{V})^{Tr}\right\}+\frac{2}{5}\left\{\nabla\vec{q}+(\nabla\vec{q})^{Tr}\right\}. \tag{VI.30}$$

The gyro-viscous momentum flux (VI.29) corresponds to the collisionless components of the momentum flux in [13] with additional terms due to the heat flux gradients, obtained by Mikhailovskii [19]. From Eqs. (VI.23) and (VI.27) it is easy to see that $p\nabla\vec{V}_p \sim \nabla\vec{q}_{\perp}^{(1)}$ so that the contributions to the collisionless momentum flux (VI.29) from $\vec{q}_{\perp}^{(1)}$ and diamagnetic velocity $\vec{V}_p$ are of the same order.

However, the direct usage of the diamagnetic velocity (VI.23) and the heat flux in (VI.27) is not practical because the corresponding components in the balance equations (VI.6)-(VI.8) contain large but divergence-free terms. These divergence-free terms can be removed in a low plasma pressure case by re-writing the corresponding terms in the continuity equation as $\nabla\cdot(n\vec{V}_p)=\nabla\cdot(n\vec{V}_D)$, where

$$\vec{V}_D = -\frac{cT}{e}\vec{B}\times\nabla\left(\frac{1}{B^2}\right), \tag{VI.31}$$

where $\vec{V}_D$ has a simple meaning of the guiding center velocity, which is different for the electrons, $\vec{V}_{De}$, and ions, $\vec{V}_{Di}$. Similarly, the contribution of the divergence-free terms to the energy balance equation (VI.8) can be removed by noticing that

$$\frac{3}{2}\vec{V}_p\cdot\nabla p+\frac{5}{2}p\nabla\cdot\vec{V}_p+\nabla\cdot\vec{q}_{\perp}^{(1)}=\frac{5}{2}\nabla\cdot(p\vec{V}_D). \tag{VI.32}$$

In addition, one can also observe that the diamagnetic contributions to the convective and gyro-viscous terms in the momentum balance equation (VI.7) are of the same order and, similarly to the particle and energy balance equations, some of these terms cancel (this is the so-called gyro-viscous cancellation of the contributions of the diamagnetic terms [24], [25], [26]). We notice that the contribution of the diamagnetic heat flux $\vec{q}_{\perp}^{(1)}$ to $\vec{\Pi}_g$ plays an important role in such cancelation [25]. However, in a non-uniform magnetic field (e.g. in a tokamak), the cancellation is not complete [26]. A somewhat similar cancelation of the collisionless terms occurs in the parallel momentum balance equation, which finally can be written as [27]:

$$mn\left\{\frac{\partial V_{\|}}{\partial t} + V_{\|}(\vec{b}\cdot\nabla)V_{\|} + (\vec{V}_E\cdot\nabla)V_{\|} + \frac{4p}{mn\Omega_B}\left[\left(\vec{b}\times\nabla\ell n(B)\right)\cdot\nabla\right]V_{\|}\right\}$$

$$-mnV_{\|}\left(\vec{V}_E\cdot\nabla\ell n(B)\right) - \frac{2V_{\|}}{\Omega_B}\left[\left(\vec{b}\times\nabla p\right)\cdot\nabla\ell n(B)\right] = en\vec{b}\cdot\vec{E} - \vec{b}\cdot\nabla p \quad \text{(VI.33)}$$

However, note that Eq. (VI.33) contains only the first-order (in $1/B$) collisionless cross-field velocities.

Overall, the contributions of the collisionless cross-field $\vec{E}\times\vec{B}$ and diamagnetic particle and energy fluxes to the particle balance equations (VI.6)-(VI.8) can play an important role, in particular for the cases where anomalous cross-field plasma transport is weak (e.g. in H-mode).

In addition to the collisionless particle, energy, and momentum fluxes, the moment equations used in the Grad approach contain also the terms associated with the collision operators (e.g. $C_n$, $\vec{C}_V$, $C_p$, etc.). These terms result in both the energy exchange and the forces (e.g. the thermal forces) between different species and provide collision-driven particle, energy, and momentum fluxes. Although the cross-field components of such fluxes are proportional to $(\rho/L_\perp)(\nu_C/\Omega_B)$ and in most cases can be ignored, the components along the magnetic field (e.g. heat flux components) can play the key roles in the balance equations (VI.6)-(VI.8). However, careful calculation of these fluxes within the framework of the Grad approach requires the implementation of the so-called 21-moment Grad approximation.

**VI.3. 21-moment Grad approximation**

As we have already mentioned above, for sufficient accuracy of the calculations of the energy and momentum fluxes $\vec{q}(\vec{r},t)$ and $\vec{\Pi}(\vec{r},t)$ as well as $\vec{C}_V$ and $C_p$, one needs also to consider the $\vec{q}^*(\vec{r},t)$ and $\vec{\Pi}^*(\vec{r},t)$ moments determined by the expressions (VI.18), (VI.19). This increases the number of independent moments that must be found. However, from the definition of $\vec{\Pi}$, recall Eq. (VI.10), it is easy to see that $\Pi_{ik} = \Pi_{ki}$ and $\Pi_{ik}\delta_{ki} = 0$, so $\vec{\Pi}$ is determined by five independent moments. The same is applicable for $\vec{\Pi}^*$. Then, allowing also for n, p (or T), $\vec{V}$, $\vec{q}$, and $\vec{q}^*$, we find that we need to determine 21 moments.

To find these moments, we consider Eq. (VI.20), (VI.21), keeping only the highest order terms. Then, for the multi-component plasma, we come to the following equations for $\vec{q}_\alpha$ and $\vec{\Pi}_\alpha$:

$$\frac{5}{2}\frac{p_\alpha}{m_\alpha}\nabla T_\alpha - \vec{q}_\alpha\times\vec{\Omega}_{B\alpha} = \sum_\beta C_{\vec{q}}^{\alpha,\beta}, \quad \text{(VI.34)}$$

$$\left(\vec{\Pi}_\alpha \times \vec{\Omega}_{B\alpha} - \vec{\Omega}_{B\alpha} \times \vec{\Pi}_\alpha\right) = p_\alpha\left(\nabla\vec{V}_\alpha + (\nabla\vec{V}_\alpha)^{Tr} - \frac{2}{3}\vec{I}\nabla\cdot\vec{V}_\alpha\right) + \frac{2}{5}\left(\nabla\vec{q}_\alpha + (\nabla\vec{q}_\alpha)^{Tr} - \frac{2}{3}\vec{I}\nabla\cdot\vec{q}_\alpha\right) - \sum_\beta C_{\vec{\Pi}}^{\alpha,\beta}, \qquad \text{(VI.35)}$$

where $C_{\vec{q}}^{\alpha,\beta}$ depends on $\vec{V}_\alpha, \vec{V}_\beta$, $\vec{q}_\alpha$, $\vec{q}_\alpha^*$, $\vec{q}_\beta$, and $\vec{q}_\beta^*$, whereas $C_{\vec{\Pi}}^{\alpha,\beta}$ depends on $\vec{\Pi}_\alpha$, $\vec{\Pi}_\alpha^*$, $\vec{\Pi}_\beta$ and $\vec{\Pi}_\beta^*$. Somewhat similar equations can be found from the evolution equations for $\vec{q}^*(\vec{r},t)$ and $\vec{\Pi}^*(\vec{r},t)$ with the collisional terms corresponding to $C_{\vec{q}^*}^{\alpha,\beta}$ and $C_{\vec{\Pi}^*}^{\alpha,\beta}$. Having found $\vec{q}_\alpha$, $\vec{q}_\alpha^*$, $\vec{q}_\beta$, and $\vec{q}_\beta^*$, one can calculate $\vec{C}_{V\alpha}$ that, along with $\vec{\Pi}_\alpha$, closes the balance equations (VI.6)-(VI.8) completely.

However, such calculations for multi-component plasma are extremely cumbersome and go beyond the scope of our consideration. The detail of such derivation can be found in [17]. Nonetheless, just for illustration, we consider here the derivation of the electron energy flux parallel to the magnetic field, $q_{\|e}$, assuming that the plasma has one kind of ions with charge Z.

Then from Eq.(VI.34) and similar equation for $\vec{q}_\alpha^*$ we find (e.g. see [17])

$$\frac{5}{2}p_e\nabla_\| T_e = \nu_{ei}\left(\frac{3}{2}p_e(V_{\|e} - V_{\|i}) - \alpha_{11}q_{\|e} - \frac{7}{2}\alpha_{12}q_{\|e}^*\right), \qquad \text{(VI.36)}$$

$$0 = -\frac{15}{4}p_e(V_{\|e} - V_{\|i}) - \alpha_{21}q_{\|e} + \frac{7}{2}\alpha_{22}q_{\|e}^*. \qquad \text{(VI.37)}$$

We notice that the terms on the right-side of these equations come from $C_{\vec{q}}^{e,i} + C_{\vec{q}}^{e,e}$ and $C_{\vec{q}^*}^{e,i} + C_{\vec{q}^*}^{e,e}$. Here $\nu_{ei} = (4/3)Z\sqrt{2\pi T_e/m_e}(e^2/T_e)^2 n_e\Lambda_C$ is the electron-ion collision frequency and $\Lambda_C$ is the Coulomb logarithm, whereas

$$\alpha_{11} = \frac{2\sqrt{2}}{5Z} + \frac{13}{10}, \quad \alpha_{12} = \alpha_{21} = \frac{1}{7}\left(\frac{3\sqrt{2}}{5Z} + \frac{69}{20}\right), \quad \text{and} \quad \alpha_{22} = \frac{9\sqrt{2}}{14Z} + \frac{433}{280}, \qquad \text{(VI.38)}$$

are the dimensionless matrix elements of the electron-ion and electron-electron Coulomb collision operators with respect to $q_{\|e}$ and $q_{\|e}^*$.

In addition, we find the following expression for the electron-ion friction force $C_{V_\|\alpha}$:

$$C_{V_\|\alpha} = -m_e n_e \nu_{ei}\left(V_{\|e} - V_{\|i} - \frac{3}{5p_e}q_{\|e} - \frac{3}{4p_e}q_{\|e}^*\right). \qquad \text{(VI.39)}$$

As one can see from Eq. (VI.39), the electron-ion friction force depends not only on the difference between the electron and ion velocities, $V_{\|e} - V_{\|i}$, but also on the electron temperature gradient $\nabla_{\|} T_e$ (this part of the friction force is called the "thermal force"). Similarly, the electron energy flux $q_{\|e}$ depends not only on the electron temperature gradient but also on the difference between the electron and ion velocities. After some algebra, from Eq. (VI.37)-(VI.39) we find:

$$q_{\|e} = -\frac{5}{2}\frac{p_e \nabla_{\|} T_e}{\nu_{ei}}\frac{\alpha_{22}}{\alpha_0^2} + \frac{3}{2} p_e (V_{\|e} - V_{\|i}) \frac{\alpha_{22} - (5/2)\alpha_{12}}{\alpha_0^2}, \qquad \text{(VI.40)}$$

where $\alpha_0^2 = \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}$.

### VI.4. The electron heat transport in a weakly collisional regime

Both the Chapman-Enskog and Grad approaches to the derivation of the closed system of the fluid equations from the kinetic ones assume a rather slow spatiotemporal variation of particle density, average velocity and temperature. In particular, the characteristic length of spatial variation of these parameters, L, (in the absence of the magnetic field or in the direction parallel to $\vec{B}$) should be larger than the mean free path of the thermal particles, $\lambda_C$. Then, as we have seen above, the distribution function $f(\vec{v}, \vec{r})$ can be expanded in the series of the integer powers of the parameter $\gamma \equiv \lambda_C / L < 1$ (here for simplicity we consider a stationary process) and the closed system of fluid equations is derived. In most cases, only linear, $\sim \gamma$, terms are held in this expansion. However, such an approach poses some questions. First, a practical one: how small the parameter $\gamma$ should be to ensure that the linear approximation describes the transport properties of the gas/plasma well? And the second, somewhat more academic question: are we sure that the distribution function does not have some terms (e.g. $\sim \exp(-C\gamma^{-r})$, where $C$ and $r$ are some positive constants) which cannot be expanded in the series of any powers of $\gamma$, but which can still be important for some range of $\gamma$ (see also [28], [16], [29]).

We start our consideration with the simpler first issue. Historically, in plasma-related applications, this issue was first raised with respect to the validity range of the Spitzer-Harm expression [30] for the electron conductive heat flux. According to Eq. 16), the heat flux $\vec{q}$ is determined by the function $\Phi_i(v') \propto \gamma$, which can be found either from expansion (VI.14) or from [30], where $\Phi_i(v')$ was obtained from the numerical solution of linearized electron kinetic equation. In [31], [32] it was pointed out that the integral expression (VI.16) for the heat flux contains high powers of the electron velocity. As a result, the main contribution to this integral comes from the velocities $v' \sim v_{cond} \sim 2 \div 3 \times v_{T_e}$. However, the magnitude of $\Phi_i(v')$ increases with increasing $v' > v_{T_e}$ (e.g. see both the expression (VI.14) and the results from [30]) and for

$v' \sim v_{cond}$ we have $\Phi(v_{cond}) \approx 10^2$ [30]. Thus, the applicability of the linearized solution for electron heat conduction, which requires $\Phi(v_{cond}) \equiv \gamma\hat{\Phi}(v_{cond}) < 1$ gives the limitation for the validity of the Spitzer-Harm expression for the electron conductive heat flux: $\gamma \lesssim 10^{-2}$. Such a severe limitation can be understood by recalling that the Coulomb mean-free path, $\lambda_v$, of a particle with velocity v scales as $\lambda_v = \lambda_T \times (v / v_{T_e})^4$. Therefore, the linear approximation for the electron conductive heat flux, which assumes that the electron distribution function is close to the Maxwellian, can only be valid if the electrons with velocities $v' \sim v_{cond} > v_{T_e}$ are collisional, $\lambda_{v_{cond}} / L \lesssim 1$, which, finally, results in $\gamma \lesssim 10^{-2}$.

However, in the edge plasmas, the inequality $\gamma \lesssim 10^{-2}$ usually does not hold and the classical expression for the electron heat flux is not applicable. A similar problem often occurs in the plasmas related to inertial confinement experiments (e.g. see [32]). For the case of $\gamma > 10^{-2}$, the electron distribution function starts to deviate significantly from the Maxwellian distribution at $v' \sim v_{cond}$. To describe this "nonlocal" effect for heat conduction along the magnetic field (in the z-direction), the expansion of the distribution function in the powers of the parameter $\gamma$ becomes impractical and an integral expression for the heat flux

$$q(z) = -\int K(z,z') \frac{\partial T_e}{\partial z'} dx', \qquad \text{(VI.41)}$$

can be considered with different kernels $K(z,z')$ which were suggested over the years (e.g. see [33], [34], [35], [36]). A comparison of the outcomes of these nonlocal models with the results of numerical simulations of the decay of a small amplitude, harmonic electron temperature perturbation is shown in Fig. (VI.1). However, in spite of the reasonable agreement of the results coming from some non-local models based on the integral expression (VI.41) and numerical simulations, in the edge fluid plasma transport codes, a much simpler approach is usually employed. It is based on the so-called "flux limiting" expression for the heat flux suggested in [38]. This expression simply majorizes the

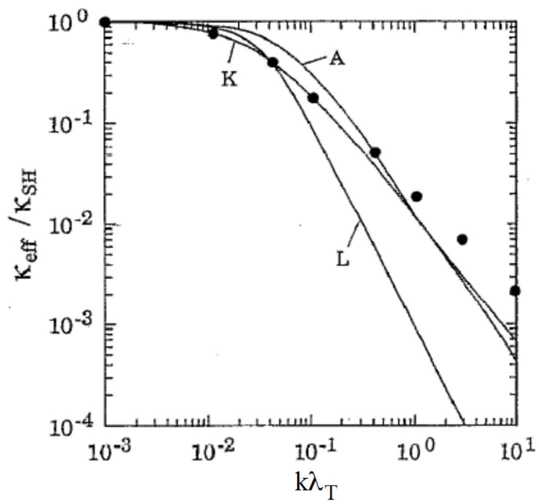


Fig. VI.1. Ratio of the effective, $\kappa_{eff}$, to the Spitzer-Härm, $\kappa_{SH}$, electron heat conductivities as a function of $k\lambda_T$, where $k$ is the wavenumber of the initial electron temperature perturbation. The filled circles are the results of numerical simulations [37], the curves A, K, and L are from the references [34], [35], and [33]. Reproduced with permission from [35], © AIP Publishing 1993.

Spitzer-Härm heat flux, $q_{SH}(z)$, by the fraction, $fr_{FS} \approx 0.2-0.4$, of the free streaming electron heat flux, $q_{FS}(z) = n(z)T_e(z)\{2T_e(z)/\pi m\}^{1/2}$, so that

$$q(x) = \frac{q_{SH}(z) fr_{FS} q_{FS}(z)}{|q_{SH}(z)| + fr_{FS} q_{FS}(z)}. \tag{VI.42}$$

As we see, such expression describes the reduction of $q(x)$ in comparison with $q_{SH}(z)$. For a monotonic temperature profile $T_e(z)$, which is the case for both the inertial fusion applications and the edge plasmas, this reduction has clear physical meaning in the high-temperature region, where, at $\gamma > 10^{-2}$, the tail of the electron distribution function is depleted because of the runaway of the weakly collisional electrons into the low-temperature region. However, in the low-temperature region, these suprathermal electrons result in the increase of $q(x)$ beyond $q_{SH}(z)$, but such an effect is not is captured by Eq. (VI.42). Nonetheless, expression (VI.42) and similar ones for ion heat conduction and viscosity along the magnetic field are often used in edge plasma transport codes.

We notice that integral expressions for the heat flux, resembling Eq. (VI.41), were suggested to emulate the effects of the Landau resonances in fluid turbulence codes (e.g. see [39] and the references therein).

Next, we discuss the issue of non-expandable terms in the distribution function and their potential impact on the electron heat flux. It is unlikely that this issue has a universal answer valid for any setting of the electron density and temperature profiles. Therefore, following [40], we consider plasma parameter profiles which resemble those typical for the SOL plasma in the high recycling conditions. In [40] it was shown that neglecting the electron-ion energy exchange, the stationary electron kinetic equation allowing for both electron-electron, $C_{ee}(f_e, f_e)$, and electron-ion scattering, $C_{ei}(f_e)$, as well as for the electric field, $E(z)$, effects:

$$v_z \frac{\partial f_e(\vec{v},z)}{\partial z} - \frac{eE(z)}{m}\frac{\partial f_e(\vec{v},z)}{\partial v_z} = C_{ee}(f_e,f_e) + C_{ei}(f_e), \tag{VI.43}$$

allows the solution in the self-similar variable $\vec{w} = \vec{v}\left[m/2T(z)\right]^{1/2}$ by using the *ansatz*

$$f_e(\vec{v},z) = NF(\vec{w})/T^{\alpha}(z), \tag{VI.44}$$

(where $T(z)$ is the effective electron temperature, $\alpha$ is an adjustable parameter, and N is the normalization constant) providing that the $T(z)$ satisfies the equation

$$\gamma = \frac{\lambda}{L} \propto T^{(\alpha-1/2)} dT/dz = \text{const.} \tag{VI.45}$$

We notice that Eq. (VI.45), gives the following relations for the electron density, $n(z) \propto T^{(3/2-\alpha)}(z)$, and the electron energy flux, $q(z) \propto T^{(3-\alpha)}(z)$. Although for $\alpha \neq 3$ $q(z)$ is not a constant, the relative magnitude of the corresponding energy source/sink,

$|dq(z)/dx|(nT\nu_{ee})^{-1} \approx (3-\alpha)\gamma$, is small for $\gamma < 1$. As a result, to maintain energy balance, a relatively small energy source/sink localized at the electron energy $\sim T$ can be added into Eq. (VI.43), which does not alter the kinetics of the energetic electrons we are mostly concerned about. Interestingly, the case $\alpha = 3$ corresponds to the electron temperature profile describing a constant electron heat flux for the Spitzer-Härm electron heat conduction coefficient $\propto T^{5/2}$.

For $\gamma << 1$ the equation for $F(\vec{w})$ was solved in [40] analytically by considering different ranges of the dimensionless velocity w and then matching the corresponding solutions (something similar was done in [41], [42] for the problem of runaway electrons). It was found that the distribution function $F(\vec{w})$ can be represented as a series in the integer powers of $\gamma$ (which is the basic assumption in both the Chapman-Enskog and Grad approaches) for $w^2 \lesssim \gamma^{-1/3}$ only, whereas for $w^2 \gtrsim \gamma^{-1/2}$, $F(\vec{w})$ is described by the following "unexpandable" expression

$$F(\vec{w}) \propto \exp\left\{-(2/3)\gamma^{-1/2}\right\}\frac{\hat{F}(\vartheta)}{w^{2\alpha}}, \tag{VI.46}$$

where $\hat{F}(\vartheta)$ is a function of the angle $\vartheta$ between the coordinate axis z and the vector $\vec{w}$ (only the terms of the highest order in the small parameter $\gamma$ are left in Eq. (VI.46)). Recalling that the main contribution to the electron conductive heat flux is due to electrons with the normalized electron velocity $\left(v/V_{T_e}\right)_{cond} = w_{cond} \approx 2 \div 3$, we find that the condition $w_{cond}^2 \lesssim \gamma^{-1/3}$ results virtually in the same limitation for $\gamma$ as it was found in [31], [32]: $\gamma \lesssim 10^{-2}$. Numerical solutions of the electron kinetic equation in self-similar variables [43] confirm the analytic results of [40]. From Eq. (VI.46) we see that the expression for the electron heat flux written in the self-similar variable w, $q = \int m w_z w^2 F(w) d\vec{w}$, diverges at large w for $\alpha \le 3$. However, for $\alpha = 3$ this divergence is logarithmically weak and can be moderated by assuming that in practice, the maximum electron energy is always limited by some value.

Thus, from the analysis of both the Spitzer-Härm solution of the electron kinetic equation and the solution of the electron kinetic equation in a self-similar variable, we find that applicability of both the Spitzer-Härm expression for electron heat conduction and the solution of kinetic electron equation in the form of expansion of the electron distribution function in integer powers of $\gamma$ are limited by relatively small $\gamma$: $\gamma \lesssim 10^{-2}$.

### VI.5. Fluid description of neutrals in edge plasmas

In edge plasmas, neutral particles (mainly atomic and molecular hydrogen) and their interactions with plasma electrons and ions play a very important role in the physics of high recycling regimes (see Ch. IX) and, in some cases, in edge plasma turbulence (see Ch. VII). Even though the most accurate description of neutral transport and neutral-plasma interactions, which can include many different atomic physics effects, is usually done with Monte-Carlo codes (see Ch. VIII), simplified fluid neutral (largely considering atomic hydrogen only) models were developed over the years and used in both analytic considerations and numerical simulations of both edge plasma transport and turbulence (e.g. see Refs. [44], [45], [46], [47], [48], [49], [50], [51], [52], and the references therein.

However, fluid neutral models developed for edge plasma studies differ significantly from the plasma fluid models. This difference is related to the fact that the neutral density, N, in the edge plasma is usually considerably lower than the plasma one, n, (unless we are dealing with a deeply detached divertor regime in fusion reactors such as ITER). As a result, the neutral-neutral collisions are much less frequent than the collisions of neutrals with plasma particles and virtually not important.

At a relatively small plasma temperature, T, the charge-exchange collisions of atomic hydrogen with protons prevail over electron impact ionization of atomic hydrogen (see Ch. II). But, in contrast to the like-like (e.g. ion-ion) collisions, such charge-exchange collisions do not result in the "Maxwellization" of the atomic hydrogen velocity distribution function, $f_a(\vec{v})$. Instead, they "drive" $f_a(\vec{v})$ toward similarity to ion (proton) distribution function, $f_i(\vec{v})$ : $f_a(\vec{v}) \propto f_i(\vec{v})$. One can easily see this from the charge-exchange collision operator

$$C_{ai}^{CX} = \int \sigma_{CX}(|\vec{v}-\vec{v}'|)\,|\vec{v}-\vec{v}'|\left(f_i(\vec{v})f_a(\vec{v}') - f_i(\vec{v}')f_a(\vec{v})\right)d\vec{v}', \qquad \text{(VI.47)}$$

where $\sigma_{CX}(|\vec{v}-\vec{v}'|)$ is the charge-exchange cross-section, which in the energy range below 10 eV can be considered constant $\approx 7\times10^{-15}\text{cm}^2$.

When the mean free path of the neutrals with respect to the charge-exchange collisions, $\lambda_{CX} \approx 1/n\sigma_{CX}$, is shorter than the characteristic scale length, $L_i$, of the variation of the ion distribution function, one can take in a "zero-order" approximation, $f_a(\vec{v}) = (N/n)f_i(\vec{v})$ and consider the mixture of the neutral atoms and plasma as a "fluid". Assuming that the ion-ion collisions "establish" the ion velocity distribution function close to a shifted Maxwellian, this becomes the starting point for the consideration of the impact of neutrals on the transport coefficients of such a fluid (e.g. see [50] for details). Since in practice $\lambda_{CX}$ is significantly larger than the ion gyro-radius, the impact of neutrals on some cross-field transport coefficients can be very significant even for the case where $N < n$. Indeed, from simple arguments, we have the following expression for the neutral diffusion coefficient: $D_N \approx (T/M)^{1/2}\lambda_{CX}$, where M is the mass of a hydrogen atom (e.g. see [50]). Then, the relative contribution, $\hat{D}_N$, of neutrals to the overall cross-field transport coefficients, such as viscosity and heat conduction, can be estimated as

$$\hat{D}_N \approx \frac{N}{n}\frac{D_N}{D_{anom}}, \qquad \text{(VI.48)}$$

where $D_{anom}$ is the anomalous cross-field plasma transport coefficient. For $D_{anom} \approx 10^4 cm^2/s$, $T \sim 10\,eV$, and $n \sim 10^{14} cm^{-3}$ we find that $\hat{D}_N \gtrsim 1$ for $N/n \gtrsim 10^{-3}$. Note that the ratio of the neutral to plasma densities for low-temperature, high recycling divertor plasma can be $\sim 10^{-1}$. However, we should keep in mind that the impact of neutrals on cross-field plasma diffusion is not described by Eq. (VI.48) since the effective displacement of the ion in the course of the elastic collision process is of the order of the ion gyro-radius. Nonetheless, the contributions of the neutrals to the momentum and heat transport can be very important for dumping the plasma flows (including the shear flow, which is an important ingredient in anomalous plasma transport, see Ch. VII) and for cooling the divertor plasma to sub-eV temperature, promoting plasma recombination effects important in the divertor plasma detachment process (see Ch. IX).

To avoid the unphysical contribution of neutrals in the edge plasma regions where $\lambda_{CX} \gtrsim L_i$, the neutral diffusive fluxes are majorized by corresponding free streaming expressions similar to Eq. (VI.42). Unfortunately, fluid description of the hydrogen molecules, which have a mean free path for the collisions with hydrogen ions much longer than that of the hydrogen atoms, strictly speaking, cannot be used for the plasma parameters of interest. In addition, vibrational excitation of molecules can play an important role in both plasma energy dissipation at low (~1 eV) temperatures and in plasma recombination processes (see Ch. II). The incorporation of vibrational excitation of molecules in fluid models would significantly complicate them.

### VI.6. Anomalous effects in edge plasma transport equations

Here we overview the basic structure of the transport equations used in edge plasma simulations. We should note that existing formulations of the transport equations often differ in various details and effects included (e.g. see [10], [11], [12]). Here we only describe the most essential elements and comment on various additional effects. Technically, either electron or ion continuity equations can be used to describe the evolution of the plasma density. Most often, the ion continuity equation is used. Keeping only the first-order terms in the ion velocity and adding an *ad hoc* anomalous density transport, one has

$$\frac{\partial n}{\partial t} + \nabla_{\|}\left(nV_{\|}\right) + \nabla\cdot\left(n\vec{V}_E\right) + \nabla\cdot(n\vec{V}_{Di}) + \nabla\cdot(\vec{\Gamma}_{an}) = S_n, \qquad \text{(VI.49)}$$

where n is the plasma density and $\vec{\Gamma}_{an}$ is the anomalous plasma density flux. In the ion continuity equation, we have neglected the second-order drift terms, such as the inertial and viscous drifts described by Eq. (VI.24). This approximation is based on the assumption that the first-order electric and diamagnetic drifts, as in Eq. (VI.23), are dominant. Note that some formulations include these higher-order drifts into the density evolution equation [10], [11].

The anomalous density flux in (VI.49) is usually defined by correlations between the density and the lowest order particle velocity due to the $\vec{E}\times\vec{B}$ drift, $\vec{\Gamma}_{an} = \left\langle \tilde{n}\tilde{\vec{V}}_E \right\rangle$, where $\tilde{n}$ and

$\tilde{\vec{V}}_E$ are the turbulence-driven fluctuating plasma density and velocity, and $\langle ... \rangle$ means statistical averaging. It is assumed here that the density fluctuation is small, $|\tilde{n}| << n$. This flux should be determined from the first-principle turbulence simulations. In transport codes, the anomalous density flux is parameterized by an empirical anomalous diffusion coefficient. In addition, besides the purely diffusive term, in some models the anomalous particle flux $\vec{\Gamma}_{an}$ includes the pinch (thermo-diffusion) terms that depend on temperature gradients (e.g. [10]):

$$\vec{\Gamma}_{an} = -D_{\perp}\nabla_{\perp}n - D_{\perp T_e} n\nabla_{\perp}\ell n(T_e) - D_{\perp T_i} n\nabla_{\perp}\ell n(T_i), \quad \text{(VI.50)}$$

where $D_{\perp}$, $D_{\perp T_e}$, and $D_{\perp T_i}$ are the anomalous "diffusivities".

The electrostatic potential $\varphi$, governing the electric field effects, is determined from conservation of the electric current $\vec{J} : \nabla \cdot \vec{J} = 0$. This equation can be written as the evolution of the generalized vorticity

$$\varpi = \frac{m_i}{B}\left(\nabla_{\perp} \cdot (n\nabla_{\perp}\varphi) + \frac{\nabla_{\perp}^2 p_i}{Ze}\right). \quad \text{(VI.51)}$$

Since the contributions of the large $\vec{E} \times \vec{B}$ drift terms of electrons and ions cancel, the second-order drift terms are usually added, which gives:

$$\begin{aligned} &\frac{\partial\varpi}{\partial t} + \vec{V}_E \cdot \nabla\varpi + \nabla \cdot \left(\frac{\vec{\Gamma}}{n}\varpi\right) = \\ &\qquad \vec{B} \cdot \nabla J_{\|} + \nabla_{\perp} \cdot \left(\mu_{\perp i}\nabla_{\perp}\varpi\right) + \nabla_{\|}\left(\mu_{\| i}\nabla_{\|}\varpi\right) + \frac{1}{e}\nabla \cdot \left\{n\left(\vec{V}_{Di} - \vec{V}_{De}\right)\right\} \end{aligned}, \quad \text{(VI.52)}$$

where $\mu_{\perp i}$ and $\mu_{\| i}$ are the anomalous viscosity coefficients and

$$J_{\|} = \sigma_{\|}\left(-\nabla_{\|}\varphi + \frac{\nabla_{\|}p_e}{en} + \frac{\alpha_T}{e}\nabla_{\|}T_e\right), \quad \text{(VI.53)}$$

is the parallel electric current, $\sigma_{\|}$ is the plasma conductivity along the magnetic field and $\alpha_T$ is the thermal force coefficient that depends on the effective ion charge $Z_{eff}$ (for $Z_{eff} = 1$, $\alpha_T = 0.71$). We omitted collisional viscosity in Eq. (VI.52) and did not include the turbulent Reynolds stress effects of the negative viscosity type, nor consider any effects of turbulent residual stresses that may result in the generation of sheared flow velocity (see Ch. VI).

The equation for plasma momentum balance can be obtained as a sum of the corresponding ion and electron equations

$$\begin{aligned} &\frac{\partial\left(nV_{\|}\right)}{\partial t} + \nabla \cdot \left\{\left(nV_{\|}\vec{b} + n\vec{V}_E + 2n\vec{V}_{Di} + \vec{\Gamma}_{an}\right)V_{\|}\right\} - nV_{\|}\vec{V}_E \cdot \ell n(B) = \\ &\qquad -\frac{\nabla_{\|}(p_e + p_i)}{m_i n} + \nabla_{\perp} \cdot \left(\mu_{\perp i}\nabla_{\perp}V_{\|}\right) + \nabla_{\|} \cdot \left(\mu_{\| i}\nabla_{\|}V_{\|}\right) + S_{iN} \end{aligned}, \quad \text{(VI.54)}$$

where $S_{iN}$ describes the impact of the plasma-neutral interactions and includes both the ion-neutral elastic collisions and neutral ionization. The anomalous terms in the momentum conservation originate from several terms involving averaging of the plasma density, parallel velocity, and the $\vec{E}\times\vec{B}$ drift fluctuations, and may also involve pressure fluctuations, e.g. see Eq (VI.33). Eq. (VI.54) includes also the collisionless first order ExB and diamagnetic fluxes shown in Eq. (VI.33).

Overall, turbulent transport is parameterized by anomalous density transport and anomalous viscosity. The turbulent momentum transport may also involve the pinch and "residual" terms, which do not depend on the velocity gradients and velocity but are rather driven by gradients of other plasma parameters.

Alternatively, only the ion momentum balance equation can be considered (see [10], [11]) although for this case, one should also solve the vorticity equation (VI.52) to find the electrostatic potential.

The electron and ion energy balance equations are most commonly written as

$$\begin{aligned}\frac{3}{2}\frac{\partial(nT_i)}{\partial t}+\nabla\cdot\left(\frac{3}{2}nT_i\vec{V}_E+\frac{5}{2}nT_i\vec{V}_{Di}+\frac{3}{2}nT_iV_{\|}\vec{b}+\frac{5}{2}\vec{\Gamma}_{an}T_i+\vec{q}_{an}^{(i)}\right)\\ +nT_i\nabla_{\|}V_{\|}-2nT_i\vec{V}_E\cdot\ell n(B)=S_{pi}\end{aligned}, \tag{VI.55}$$

$$\begin{aligned}\frac{3}{2}\frac{\partial(nT_e)}{\partial t}+\nabla\cdot\left\{\frac{3}{2}nT_e\vec{V}_E+\frac{5}{2}nT_e\vec{V}_{De}+\frac{3}{2}nT_e\left(V_{\|}-\frac{J_{\|}}{en}\right)\vec{b}+\frac{5}{2}\vec{\Gamma}_{an}T_e+\vec{q}_{an}^{(e)}\right\}\\ +nT_e\nabla_{\|}\left(V_{\|}-\frac{J_{\|}}{en}\right)-2nT_e\vec{V}_E\cdot\ell n(B)=S_{pe}\end{aligned}, \tag{VI.56}$$

where $\vec{q}_{an}^{(e,i)}=-n\chi_{\perp}^{(e,i)}\nabla_{\perp}T_{(e,i)}$ describe anomalous electron and ion heat conduction determined by the anomalous heat diffusivities, $\chi_{\perp}^{(e,i)}$; $S_{pe}$ and $S_{pi}$ are the electron and ion energy sinks/source terms describing the Joule heating, electron-ion energy exchange, ion and electron interactions with neutrals, etc.

Strictly speaking, equations (VI.49)-(VI.56) should be accompanied by corresponding equations for the impurities, which are ubiquitous in edge plasmas. However, the impurity equations are very cumbersome (e.g. see [17]) and their consideration goes beyond the scope of this chapter.

We note, however, that in many cases, the analysis of the experimental data suggests that the anomalous convective cross-field energy transport should enter with the coefficient 3/2 rather than with 5/2 as it is written in Eq (VI.55) and (VI.56). Such a conclusion is also supported by some theoretical arguments valid for the case where the plasma parameters can be separated into the mean and the small, turbulence-driven fluctuating parts (see [53], [54]) when the anomalous particle and heat fluxes can be defined as $\vec{\Gamma}_{an}=\left\langle\tilde{n}\tilde{\vec{V}}_E\right\rangle$ and $\vec{q}_{an}=(3/2)n\left\langle\tilde{T}\tilde{\vec{V}}_E\right\rangle$. In this case, one more term, $-2\left\{T\vec{\Gamma}_{an}+(2/3)\vec{q}_{an}\right\}\cdot\ell n(B)$, should be added on the left-hand side of the pressure balance equation Eq. (VI.55), (VI.56).

**Conclusions for Chapter VI**

As of today, modeling of edge plasma transport, which incorporates the particle, momentum, and energy fluxes, the atomic physics, the plasma-wall interactions, etc., relies on fluid plasma models. Although such models have some issues with their applicability (e.g. an impact of nonlocal effects on electron heat transport) and use a crude and, actually, *ad hoc* description of anomalous cross-field transport, they reproduce many features observed in experiments (see Ch. VIII, IX).

**References for Chapter VI**


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