Secondary instabilities in the dynamics of the Farley-Buneman fluctuations

A. I. Smolyakov, A. V. Koustov and R. A. Makarevitch

Short title: SECONDARY FARLEY-BUNEMAN WAVES

Abstract. Nonlinear dynamics of the Farley-Buneman modes is investigated. It is shown that, in the nonlinear regime, there exists a secondary nonlinear instability leading to excitation of low frequency, long wavelength modes that corresponds to the inverse energy cascade toward the longer wavelengths. The growth rate of the secondary instability is proportional to the squared amplitude of primary electron density fluctuations.

Department of Physics and Engineering Physics, University of
Saskatchewan, Saskatoon, S7N 5E2 Canada

1. Introduction

The Farley-Buneman (F-B) instability [*Farley*, l963; *Buneman*, 1963] is one of the mechanisms for excitation of E region electrojet irregularities. These irregularities have been intensively studied using VHF coherent radars [*Fejer and Kelley*, 1980; *Sahr and Fejer*, 1996]. The theoretical framework for the linear theory of the F-B instability is well established [e.g., *Dimant and Sudan*, 1995; *Sahr and Fejer*, 1996; *Kissack and St-Maurice*, 2000] while there remains a number of issues within the non-linear theory that are now actively investigated [e.g., *Hamza and St-Maurice*, 1993; *Sahr and Farley*, 1995, *Sudan et al.*, 1997; *Dimant*, 1999]. Analytical studies have been complemented by computer simulations [*Ronchi et al.*, 1991; *Schlegel and Thiemann*, 1994; *Janhunen,* 1994; *Oppenheim et al.*, 1996]. Despite significant progress in understanding the F-B instability evolution, there are still significant discrepancies between theoretical predictions and observations, as reviewed by *Sahr and Fejer* [1996]. This urges further development of the theory for this instability.

One of the features identified in the equatorial VHF observations is that coherent echoes (scatter from meter-scale irregularities) are often modulated by the low frequency, long wavelength perturbations [*Farley et al.*, 1994]. These measurements suggest that the evolution of the F-B instability at equatorial latitudes may be controlled by the large-scale dynamics [*Farley et al.*, 1994] rather than by the interactions among the short-wave modes themselves even though the F-B instability is most efficiently excited at short (meter) wavelengths.

Ronchi et al. [1991] have investigated the evolution of large-scale structures and their role in the dynamics of the meter scale irregularities for the case of the gradient-drift instability. For the pure F-B instability computer simulations by *Oppenheim et al.* [1996] have shown that excitation of the smaller scale secondary waves plays an important role in the nonlinear saturation of this instability. This result means that there is an energy flow toward smaller scales, i.e. in the standard direction of the energy flow (forward cascade). On the other hand, one might think that the small-scale ionospheric turbulence can provide energy for support of large-scale irregularities. Since F-B waves are excited only at scales shorter than ∼100 m [*Schelgel and St-Maurice*, 1983], their support of significantly larger scale fluctuations would require an energy flow toward larger scales/smaller wavenumbers, i.e. an inverse cascade. Migration of energy to larger scales in the process of the electrojet instability evolution has been noticed by *Ronchi et al.* [1991]. One should mention that in the past, the growth of large-scale perturbations has been addressed within the framework of nonlocal linear theory [e.g., *Kudeki et al.*, 1982; *Huba and Lee*, 1983; *Fu et al.*, 1986].

In this paper we explore a possibility of energy transfer from small to large scales in the course of the F-B instability. In this respect, *Sharma and Kaw* [1986] considered modification of the large-scale gradient-drift (G-D) modes in a presence of the small-scale F-B waves and predicted that the effect can be significant under some conditions. Our goal is somewhat similar; we study the dynamics of long wavelength perturbations that are coupled to and modified by the primary (small-scale) F-B fluctuations. A related problem was also considered by *Sahr and Farley* [1995], who analyzed excitation of secondary waves via the three-wave coupling mechanism.

The paper is organized as follows. In Section 2 we review basic nonlinear equations describing the F-B modes. Then (Section 3) we derive a wave kinetic equation for the background fluctuations in a presence of low frequency perturbations. Coupled dynamics of these perturbations and the background fluctuations is considered in Section 4, where we derive the final nonlinear dispersion equation. This equation is then analyzed in Section 5.

2. Basic equations

We use the fluid two-dimensional equations for a collisional plasma [*Sudan et al.*, 1973]. We first review basic assumptions and present the main equations following *Sudan et al.* [1973] (see also *Sudan*, 1983; *Sahr*, 1990; *Hamza and St-Maurice*, 1993; *Sahr and Farley*, 1995).

We start from the ion continuity equation for the charge density n and velocity $\mathbf{V}_i$ in a form

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}_i) = 0. \tag{1}$$

The ion fluid velocity $\mathbf{V}_i$ can be found from the ion momentum equation. We assume unmagnetized ions since the collision frequency of ions, ν_i, is larger than the ion cyclotron frequency ω_{ci}, $\nu_i \gg \omega_{ci}$. Then, the ion momentum equation is

$$m_i \frac{d\mathbf{V}_i}{dt} = e\mathbf{E} - \frac{T_i}{n_0}\nabla n - \nu_i m_i \mathbf{V}_i. \tag{2}$$

Here m_i and e are the ion mass and charge, $\mathbf{E}$ is the total electric field, T_i is the ion temperature. For the F-B modes, the electron fluid velocity is much larger than the ion velocity, $V_e \gg V_i$. As a result, the main source of nonlinear effects is contained in the electron momentum equation so that the ion equation (2) can be linearized. After linearization of equations (1) and (2) and exclusion of the ion velocity $\mathbf{V}_i$, one obtains

$$\left(\frac{\partial}{\partial t} + \nu_i\right)\frac{\partial}{\partial t}\frac{\tilde{n}}{n_0} + \nabla\cdot\left(\frac{e}{m_i}\mathbf{E} - \frac{T_i}{m_i n_0}\nabla\tilde{n}\right) = 0. \tag{3}$$

Here we assumed that the total plasma density n is a sum of the equilibrium n_0 and perturbed $\tilde{n}$ quantities, and the plasma background is uniform, $\nabla n_0 = 0$.

Electrons are magnetized, because the electron cyclotron frequency ω_{ce} is larger than the electron collision frequency, $\omega_{ce} \gg \nu_e$. For the low frequency oscillations $\omega \ll \nu_e$ the electron momentum equation is

$$0 = -e\mathbf{E} - \frac{e}{c}\mathbf{V}_e \times \mathbf{B} - \frac{T_e}{n_0}\nabla\tilde{n} - \nu_e m_e \mathbf{V}_e. \tag{4}$$

Excluding the electric field from (3) and (4), one obtains

$$\left(\left(\frac{\partial}{\partial t} + \nu_i\right)\frac{\partial}{\partial t} - C_s^2\nabla^2\right)\frac{\tilde{n}}{n_0} - \frac{\omega_{ce}\omega_{ci}}{\nu_e}\nabla\cdot\mathbf{V}_e = 0, \tag{5}$$

where $C_s = \sqrt{(T_e + T_i)/m_i}$ is the ion-acoustic speed of the medium. Excluding $\nabla\cdot\mathbf{V}_e$ from (5) with the help of the electron continuity equation, we obtain the basic nonlinear equation [*Sudan et al.*, 1973; *Sudan,* 1983; *Sudan and Keskinnen*, 1977; *Hamza and St-Maurice*, 1993; *Sahr,* 1990; *Sahr and Farley*, 1995]

$$\frac{\partial}{\partial t}\tilde{n} + \frac{1}{1+\psi}\mathbf{V}_0 \cdot \nabla \tilde{n} + \frac{1}{1+\psi}\mathbf{V}_E \cdot \nabla \tilde{n} + \frac{1}{\nu_i}\frac{\psi}{1+\psi}\left(\frac{\partial^2}{\partial t^2} - C_s^2 \nabla^2\right)\tilde{n} = 0. \tag{6}$$

Here

$$\psi = \frac{\nu_e \nu_i}{\omega_{ce}\omega_{ci}}, \tag{7}$$

and $\mathbf{V}_0 = c\mathbf{E}_0 \times \mathbf{B}/\mathbf{B}^2$ is the electric drift in the equilibrium electric and magnetic fields $\mathbf{E}_0$ and $\mathbf{B}$, and $\mathbf{V}_E = -c\nabla\phi \times \mathbf{B}/\mathbf{B}^2$ is the drift velocity in the perturbed electric field $E = -\nabla\phi$ (ϕ is electrostatic potential). In the nonlinear term (the third term in (6)) we retain only the lowest order electron velocity given by the electric drift, $\mathbf{V}_e \simeq \mathbf{V}_E$.

Making the Fourier transformation in space, we rewrite equation (6) in a form

$$\frac{\partial}{\partial t}n_k + i\omega_k n_k + \int d\mathbf{k}_1 V_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} n_{\mathbf{k}_1} n_{\mathbf{k}-\mathbf{k}_1} = 0. \tag{8}$$

Here the nonlinear coupling matrix is given by

$$V_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} = -\frac{1}{1+\psi}\frac{\nu_i}{\omega_{ci}}\frac{i\omega_{k_1}}{n_0 k_1^2}\mathbf{b} \times \mathbf{k}_1 \cdot (\mathbf{k} - \mathbf{k}_1), \tag{9}$$

where $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the equilibrium magnetic field.

By neglecting the nonlinear terms, one obtains from (6) the linear eigenfrequency ω_k,

$$\omega_k = \omega_{rk} + i\gamma_k, \tag{10}$$

where the real part of the frequency is given by an expression

$$\omega_{rk} = \frac{\mathbf{k} \cdot \mathbf{V}_0}{1+\psi}, \tag{11}$$

and the instability growth rate is defined by

$$\gamma_k = \frac{1}{\nu_i}\frac{\psi}{1+\psi}\left(\omega_{rk}^2 - k^2C_s^2\right). \tag{12}$$

The instability occurs for fluctuations with

$$\omega_{rk}^2 > k^2C_s^2, \tag{13}$$

that defines the threshold for the phase velocity on unstable waves.

It is convenient to write equation (6) by using a new variable F introduced by the relation [*Sudan et al.*, 1997].

$$\frac{\partial F}{\partial y} = \phi, \tag{14}$$

where ϕ is electrostatic potential. From the ion continuity equation (1) one gets, in the lowest order, the relation between the density and potential fluctuations

$$k^2\phi_k = i\omega_r k\frac{m_i\nu_i}{en_0}n_k = i\frac{\mathbf{k}\cdot\mathbf{V}_0}{1+\psi}\frac{m_i\nu_i}{en_0}n_k. \tag{15}$$

In real space, it takes the form

$$-\nabla^2\phi = \frac{m_i\nu_i}{en_0}\frac{1}{1+\psi}\mathbf{V}_0\cdot\frac{\partial}{\partial\mathbf{x}}\tilde{n}, \tag{16}$$

where $\mathbf{V}_0 = V_{0y}\mathbf{e}_y$ is the electron drift velocity (assumed along y-axis).

Then we can write

$$-\nabla^2 F = \frac{m_i\nu_i}{en_0}\frac{1}{1+\psi}V_{0y}\tilde{n}. \tag{17}$$

With this new variable, the nonlinear equation (6) can be written as

$$\frac{\partial}{\partial t}\nabla^2 F + \widehat{L}\nabla^2 F + \frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial F}{\partial y}\cdot\nabla\nabla^2 F \qquad = 0. \tag{18}$$

The linear operator $\widehat{L}$ describes the real part of the wave frequency and the growth rate. In Fourier representation, this operator takes the form

$$\widehat{L} \rightarrow i\left(\omega_{rk} + i\gamma_k\right). \tag{19}$$

3. Multiple scale separation and wave kinetic equation

We assume that the background primary fluctuations have reached their steady-state. We do not consider in details how the saturation was achieved. In general, the steady-state is established as a result of a competition between the linear instability and nonlinear damping due to mode interaction [*Hamza and St-Maurice,* 1993b].

We consider now a perturbation of the background equilibrium turbulence due to the secondary, low frequency, long wavelength modes. Thus, we assume the scale separation between the background fluctuations and the secondary waves. This approach allows us to use averaging over the small-scale fluctuations.

We present fluctuations as a sum of fast and slow parts

$$F = \overline{F} + \widetilde{F}. \tag{20}$$

The fast part $\widetilde{F} = \widetilde{F}(\mathbf{x}, t, \mathbf{X}, T)$ depends on fast $(\mathbf{x}, t)$ and slow $(\mathbf{X}, T)$ variables while the slow part $\overline{F} = \overline{F}(\mathbf{X}, T)$ is a function of variables $(\mathbf{X}, T)$ only; the slow variables are formally introduced by $\mathbf{X} = \boldsymbol{\varepsilon}\mathbf{x}$, $T = \varepsilon t$, where $\varepsilon \ll 1$ is the small parameter of the scale separation.

Equations for $\overline{F}$ and $\widetilde{F}$ are derived from the nonlinear equation (18). The equation for the evolution of the mean function $\overline{F}$ is obtained from (18) by averaging over small scale fluctuations

$$\frac{\partial}{\partial t}\nabla^2\overline{F} + \hat{L}\nabla^2\overline{F} + \frac{c}{B_0(1+\psi)}\frac{\partial}{\partial \mathbf{X}}\cdot\overline{\left(\mathbf{b}\times\nabla\frac{\partial\widetilde{F}}{\partial y}\nabla^2\widetilde{F}\right)} = 0. \tag{21}$$

The equation for the small-scale component is

$$\frac{\partial}{\partial t}\nabla^2\widetilde{F} + \hat{L}\nabla^2\widetilde{F} + \frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial Y}\cdot\nabla\nabla^2\widetilde{F}$$
$$+\frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\widetilde{F}}{\partial y}\cdot\nabla\nabla^2\widetilde{F} = 0 \tag{22}$$

The last term in this formula describes the self-interaction of small-scale perturbations. This self-interaction is important to establish the equilibrium steady state.

The interaction of small-scale fluctuations with the mean flow is described by the third term in (22). In general, the coupling of small-scale fluctuations to the mean flow can be described by the kinetic equation for wave packets, which is the conservation law for the wave action invariant N_k. The standard form [*Kadomtsev,* 1964; *Vedenov* et al., 1967] for such an equation is

$$\frac{\partial N_k}{\partial T} + \frac{\partial\omega'_{rk}}{\partial\mathbf{k}}\cdot\frac{\partial N_k}{\partial\mathbf{X}} - \frac{\partial\omega'_{rk}}{\partial\mathbf{X}}\cdot\frac{\partial N_k}{\partial\mathbf{k}} = S_k, \tag{23}$$

where ω_{rk} is the real part of the wave eigenfrequency, and $N_k = N_k(\mathbf{X},T)$ is the generalized wave action density, which is a slow function in time and space.

The coupling of the intensity of small-scale fluctuations N_k with the slow component $\overline{F}$ is provided via the variations of the wave eigenfrequency by the slow component of the electric field, $\omega'_{rk} = \omega'_{rk}(\mathbf{X},T)$, where ω'_{rk} is the wave frequency taking into account the contribution of the slow component. From equation (6) one can easily find that

$$\omega'_{rk} = \omega_{rk} + \frac{c}{B_0}\frac{1}{1+\psi}\mathbf{b}\times\nabla\overline{\phi} = \omega_{rk} + \frac{c}{B_0}\frac{1}{1+\psi}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial Y}, \tag{24}$$

where ω_{kr} is the wave eigenfrequency in the stationary plasma.

As shown in the Appendix, the generalized wave action N_k in our case is given by the expression $N_k = k^4 \left|\widetilde{F}_k\right|^2$, so that the wave kinetic equation for the wave packets in the presence of a slow, long wavelength component takes the form

$$\begin{aligned}
&\frac{\partial}{\partial T}\left(k^4\left|\widetilde{F}_k\right|^2\right) \\
&+\frac{\partial}{\partial \mathbf{k}}\left(\omega_{rk}+\frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial Y}\cdot\mathbf{k}\right)\cdot\frac{\partial}{\partial \mathbf{X}}\left(k^4\left|\widetilde{F}_k\right|^2\right) \\
&-\frac{\partial}{\partial \mathbf{X}}\left(\omega_{rk}+\frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial Y}\cdot\mathbf{k}\right)\cdot\frac{\partial}{\partial \mathbf{k}}\left(k^4\left|\widetilde{F}_k\right|^2\right) = S_k. \qquad (25)
\end{aligned}$$

The right hand side of equation (25) describes the wave growth due to the linear instability and the nonlinear wave damping due to the self-interaction of small scales. In the equilibrium, the self-interaction of small scales balances the linear growth rate. This balance can symbolically be represented as $S_k = \gamma_k N_k - \triangle\omega_k N_k^2$, where γ_k is the linear growth rate, and $\triangle\omega_k N_k^2$ describes the nonlinear self-interaction leading to the damping of small-scale perturbations, $\triangle\omega_k$ is the nonlinear decrement. In the stationary state $S_k \to 0$.

4. Coupled dynamics of short- and long-wavelength components and secondary instability

Equations (21) and (25) describe coupled dynamics of the slow varying field $\overline{F}(\mathbf{X},T)$ and the wave action density $N_k(\mathbf{X},T)$. Now we can investigate the stability of this system with respect to slow deviations from the stationary turbulent state. We assume that the steady state fluctuation spectrum N_k^0 is maintained through competition of the linear growth and nonlinear damping due to interactions among small scales (this balance is described by the last term in equation (25), which is thus zero in the equilibrium, $S_k = 0$). We assume that there is a small deviation δN_k from this equilibrium, $N_k = N_k^0 + \delta N_k$. This deviation is related to the slow perturbation in the F field which is thus a slow field $\overline{F}$. The system is linear with respect to δN_k and $\overline{F}$; one can use the Fourier representation to analyze it

$$\left(\delta N_k,\ \overline{F}\right) \propto \exp(-i\Omega T + i\mathbf{q}\cdot\mathbf{X}). \tag{26}$$

We consider small perturbations with a wave vector $\mathbf{q}$ along the equilibrium flow $\mathbf{V}_0 = V_{0y}\widehat{\mathbf{y}}$, $\mathbf{q} = q\widehat{\mathbf{y}}$. Linearizing equations (25) with respect to δN_k we obtain

$$-i\left(\Omega - qV_{gr}\right)\delta N_k - iq^3 \frac{c}{B_0(1+\psi)}\overline{F}k_x\frac{\partial N_k^0}{\partial k_y} = 0, \tag{27}$$

where $V_{gr} = \partial\omega_{rk}/\partial k_y$ is the group velocity of the primary (small-scale) F-B waves. From (21) we find

$$i\Omega q^2\overline{F} + \frac{c}{B_0(1+\psi)}iq\sum k_x k_y k^2\left|\widetilde{F}_k\right|^2 = 0. \tag{28}$$

Perturbations of the spectrum $\left|\widetilde{F}_k\right|^2$ can be related to the wave action perturbations by the relation

$$k_x k_y k^2\left|\widetilde{F}_k\right|^2 = \frac{k_x k_y}{k^2}\delta N_k. \tag{29}$$

The sum in (28) is taken over the spectrum of small-scale fluctuations. It occurs as a result of averaging of the last term in equation (21).

Excluding $\overline{F}$ and δN_k from equations (27) and (28), we obtain the dispersion relation

$$\Omega\left(\Omega - qV_{gr}\right) + \Gamma_0^2 = 0. \tag{30}$$

The nonlinear growth rate Γ_0 is defined by

$$\Gamma_0^2 = -\left(\frac{c}{B_0(1+\psi)}\right)^2 q^2 \sum \frac{k_x^2 k_y}{k^2}\frac{\partial N_k^0}{\partial k_y}. \tag{31}$$

As follows from the analysis below, the parameter Γ_0^2 defines the growth rate of the long wavelength instability. To evaluate the magnitude of this parameter, we use an estimate from (17)

$$k^2 F \simeq \frac{m_i \nu_i}{e n_0}\, V_{0y}\widetilde{n}. \tag{32}$$

Using also $N_k = k^4 \left|\widetilde{F}_k\right|^2$ and assuming a roughly isotropic spectrum of the turbulence, we obtain for Γ_0^2 the following order of magnitude estimate in terms of density fluctuations

$$\Gamma_0^2 = q^2 \left(\frac{\nu_i}{\omega_{ci}}\right)^2 V_{0y}^2 \left(\frac{\widetilde{n}}{n_0}\right)^2. \tag{33}$$

Note that $\frac{\partial N_k^0}{\partial k_y} < 0$ is required for the instability. This is a typical condition for the saturated turbulence.

5. Analysis of the nonlinear dispersion equation

In this section we analyze dispersion equation (30). One can see that nonlinear effects of small-scale fluctuations given by the term Γ_0^2 are destabilizing while the effects of finite packet width (the term with group velocity qV_{gr}) are stabilizing. The latter term provides a finite amplitude threshold for the instability. From (30) we obtain

$$\Omega = \frac{qV_{gr}}{2} \pm \sqrt{\frac{(qV_{gr})^2}{4} - \Gamma_0^2} \tag{34}$$

Nonlinear instability occurs for

$$\Gamma_0^2 > \frac{(qV_{gr})^2}{4} \tag{35}$$

or

$$\left(\frac{\nu_i}{\omega_{ci}}\right)^2 \left(\frac{\widetilde{n}}{n_0}\right)^2 > \frac{1}{4}. \tag{36}$$

In the limit $\Gamma_0 \to 0$, the dispersion equation (34) simply describes the oscillating wave packets that move with the group velocity of the F-B mode.

As a matter of fact, in (34) we have neglected the linear dynamics of the secondary mode, that is given by the linear operator $\widehat{L}$ in equation (21). Restoring these effects, we obtain for the general case

$$\left(\Omega - \omega_{rq} - i\gamma_q\right)\left(\Omega - qV_{gr}\right) + \Gamma_0^2 = 0, \tag{37}$$

where $\omega_{rq} + i\gamma_q$ is the real frequency and growth rate of the F-B fluctuations with the wave vector q. They are given by expressions (11) and (12) with the wave number q substituted for k.

Nonlinear dispersion equation (37) describes two branches of fluctuations: the regular F-B mode with a wave number q and $\Omega = \omega_{rq} + i\gamma_q$ and the modes with $\Omega = qV_{gr}$ that correspond to the long wave-length modulations of the background turbulence moving with the group velocity of the F-B mode. In the presence of nonlinear effects, these two modes are coupled via the nonlinear interaction described by the Γ_0^2 term. The result is the nonlinear instability that produces long wavelength perturbations.

A simple analysis can be made taking into account that $\omega_{rq} \simeq qV_{gr}$. Then the dispersion equation (37) can be rewritten as

$$\left(\Omega' - i\gamma_q\right)\Omega' + \Gamma_0^2 = 0, \tag{38}$$

where $\Omega' \equiv \Omega - qV_{gr}$ is the mode frequency in the reference frame moving with the group velocity. The dispersion equation (38) yields

$$\Omega' = \frac{i\gamma_q}{2} \pm \sqrt{-\frac{\gamma_q^2}{4} - \Gamma_0^2}. \tag{39}$$

One remarkable property of this expression is that the instability occurs for any sign of γ_q. This means that (due to nonlinear effects for the longer wavelengths) there is no ion-acoustic threshold for the secondary instability development contrary to what is known for the primary F-B instability as one can infer from equation (13). This conclusion also holds in a more general case where $\omega_{rq} \neq qV_{gr}$. In the latter case we have

$$\Omega = \frac{qV_{gr} + \omega_{rq}}{2} + \frac{i\gamma_q}{2} \pm \frac{1}{2}\sqrt{D}, \tag{40}$$

where

$$D = (i\gamma_q + qV_{gr} + \omega_{rq})^2 - 4(i\gamma_q qV_{gr} + \Gamma_0^2 + \omega_{rq} qV_{gr}). \tag{41}$$

The imaginary part of the frequency is given by the following expression

$$Im\ \Omega = \frac{\gamma_q}{2} \pm \frac{1}{2}\sqrt{\frac{r-a}{2}}.$$

Here

$$r^2 = ((qV_{gr} - \omega_{rq})^2 - \gamma_q^2 - 4\Gamma_0^2)^2 + 4(qV_{gr} - \omega_{rq})^2\gamma_q^2,$$

and

$$a = (qV_{gr} - \omega_{rq})^2 - \gamma_q^2 - 4\Gamma_0^2. \tag{42}$$

After simple algebra, one can show that

$$r > \gamma_q^2 + (qV_{gr} - \omega_{rq})^2 - 4\Gamma_0^2, \tag{43}$$

and hence

$$(r-a)/2 > \gamma_q^2.$$

The latter inequality means that for any sign of γ_q, one of the two roots

$$Im\,\Omega = \frac{\gamma_q}{2} \pm \frac{1}{2}\sqrt{\frac{r-a}{2}}$$

is positive that corresponds to the instability. For stable F-B modes with $\gamma_d < 0$, the growth rate is proportional to the amplitude of the background turbulence. From (39), in the regime of $\Gamma_0^2 < \gamma_q^2/4$, one can find

$$Im\,\Omega \simeq -\Gamma_0^2/\gamma_q. \tag{44}$$

6. Discussion and Conclusions

In the present paper we have considered the dynamics of the long wavelength perturbations of the turbulent background made up by small-scale F-B (primary) waves. We have shown that the F-B waves that reached the equilibrium state are unstable with respect to the long wavelength perturbations propagating along the background plasma flow with the velocity approximately equal to the group velocity of the F-B modes. The growth of the large-scale structures can be interpreted as the inverse energy cascade with the energy flow toward longer wavelengths.

The mechanism of the secondary instability considered in our paper is related to the wave coupling between the small-scale fluctuations and the long-wavelength modes (a parametric process somewhat similar to that considered by *Sharma and Kaw* [1986]). The secondary mode generation is related to the nonlinear terms in the electron continuity equation. The main source of the nonlinearity is the advection of the perturbed plasma density by the $\mathbf{E} \times \mathbf{B}$ drift [*Oppenheim*, 1996]. The small scales are affected by the induced slow mode due to refraction in the shear $\mathbf{E} \times \mathbf{B}$ flow associated with the slow mode. Essentially, the source of this refraction is the modification of the phase velocity of small-scale fluctuations by the electric field of the slow mode.

We used the wave kinetic equation to describe the modulations of the background small-scale turbulence. It should be noted that the primary small-scale fluctuations must be excited before the secondary instability takes place. We did not consider the mechanism for the primary F-B wave saturation. It was assumed that primary wave reach their saturated state and that then they interact with secondary large-scale modes. This approach is similar to that used in the theory of zonal flow generation in geostrophic fluids such as the atmospheres of rotating planets [*Busse and Or*, 1986; *Smolyakov et al.*, 2000]. An interesting feature of the secondary instability considered in our work is that it does not have the ion-acoustic threshold typical for the primary F-B modes. The growth rate of secondary large-scale modes is determined by the intensity of the primary fluctuations. Certainly, the growth of the secondary instability will be evetually slowed down, and the saturation of the whole primary-secondary wave system will be reached due to the back influence of the large-scale modes on the small scales and the nonlinear interaction between the secondary modes. Quantitative analysis of these processes requires a nonlinear theory for the secondary modes that is beyond the scope of the present study.

Our approach to the analysis of the F-B instability differs from previous studies in several aspects. The major difference is that we assume the energy transfer to the longer perturbations, outside the linear instability range for the primary F-B waves (to scales, say, of a kilometer size). Traditionally, the energy transfer to shorter scales was considered, starting from the work by *Sudan et al.* [1973], see also *Moiseev et al.,* 2000). Our analysis is closely related to the work by *Sharma and Kaw* [1986] who have suggested that the dynamics of secondary G-D waves can be strongly modified in the presence of small-scale F-B fluctuations. These authors have also anticipated a possiblity of nonlinear instability supported by the primary fluctuations. Contrary to this work, we considered perturbations propagating along the background flow, the most favorable direction for the F-B instability. Though technically different, our analysis is somewhat similar to that of *Sahr and Farley* [1995]. Analogously to that work, the long wavelength modes in our model originate from three-wave coupling processes. These long wavelength modes are excited by "beating" of small-scale fluctuations.

Implication of our scenario of the F-B instability evolution is that the long wavelength secondary waves (at scales that cannot be excited linearly, namely more than $\sim$100 m) should be observed along the electrojet. Since the group velocity of the F-B fluctuations is close to their phase velocity, the nonlinearly generated large-scale modes should move with the velocity of the background fluctuations. These predictions seem to agree with radar observations at equatorial latitudes [*Farley et al.*, 1994]. We are not aware of any experiments that would support this scenario of the large-scale wave excitation in the auroral ionosphere.

Appendix A: Wave kinetic equation

Dynamics of a wave packet in an inhomogeneous plasma is described by the wave kinetic equation for the wave action density [*Kadomtsev*, 1964; *Vedenov* et al., 1967]. Here, we review the derivation of the wave kinetic equation that considers the interaction of small-scale fluctuations with a slow-varying mean flow, following to *Smolyakov and Diamond* [1999] (see also *Dubrulle et al.,* 1997). We consider a generic equation for fluctuating field F_k in a form

$$\frac{\partial F_k}{\partial t} + i\omega_k F_k + \int d^2p M_{p,k-p} F_p F_{k-p} = 0, \tag{A1}$$

where $\omega_k = \omega(k)$ is the frequency of the linear mode with a wave vector k (the frequency may include an imaginary part corresponding to the wave growth or decay). From equation (22) we find the coupling matrix $M_{p,k-p}$

$$\begin{aligned} M_{p,k-p} \;\; = \;\; & -\frac{ic}{B_0(1+\psi)} \mathbf{b} \times \mathbf{p} \cdot (\mathbf{k}-\mathbf{p}) \frac{p_y(\mathbf{k}-\mathbf{p})^2}{k^2} \\ & -\frac{ic}{B_0(1+\psi)} \mathbf{b} \times \mathbf{p} \cdot (\mathbf{k}-\mathbf{p}) \frac{p^2(k-p)_y}{k^2}. \end{aligned} \tag{A2}$$

To keep track of the scale separation explicitly it is convenient to introduce new notations for the large-scale $F_k^<$ and small-scale $F_k^>$ components; $F_k^< = 0$ for $\mid \mathbf{k} \mid > \mathbf{k}_\varepsilon$, and $F_k^> = 0$ for $\mid \mathbf{k} \mid < \mathbf{k}_\varepsilon$, where $\mathbf{k}_\varepsilon = \mathbf{k}_0 \varepsilon$ defines a boundary of the scale separation, $\varepsilon \ll 1$ is a scale separation parameter, and $\mathbf{k}_0$ is the characteristic wave number; $F_k^<$ corresponds to $\overline{F}$ in the real space, $F_k^< \rightarrow \overline{F}$.

For a near stationary case of an anisotropic drift-wave type turbulence the interaction of small scales with a long wave-length component is dominant [*Balk et al.*, 1990], so the self-interaction of small-scale fields can be neglected. Such interaction is important however to establish the stationary spectrum that is formed by balancing the linear growth rate with the nonlinear damping. These effects are described by the right hand side of equation (23).

Retaining only the dominant interaction term we write from (A1) the following equation for the small-scale fluctuations

$$\frac{\partial F_k^{>}}{\partial t} + i\omega_k F_k^{>} + \int d^2p M_{p,k-p} F_p^{<} F_{k-p}^{>} = 0. \tag{A3}$$

To derive the equation for the evolution of the wave spectrum we multiply equation (A3) by $F_{k'}^{>}$ and then add it with a similar equation obtained by reversing k and k', yielding

$$\begin{aligned} &\frac{\partial}{\partial t}\left(F_k^{>} F_{k'}^{>}\right) + i\left(\omega_k + \omega_{k'}\right)\ F_k^{>} F_{k'}^{>} \\ &+F_{k'}^{>} \int d^2p M_{p,k-p} F_p^{<} F_{k-p}^{>} + F_k^{>} \int d^2p M_{p,k'-p} F_p^{<} F_{k'-p}^{>} = 0. \end{aligned} \tag{A4}$$

The small-scale turbulence is described by the spectral function (Wigner function) $I_k(\mathbf{x}, t)$ and defined as follows

$$\int d^2q \left\langle F_{-k+q}^{>} F_k^{>} \right\rangle \exp(i\mathbf{q}\cdot\mathbf{x}) = I_k(\mathbf{x}, t). \tag{A5}$$

Hereafter, the dependence on fast variables is suppressed, and ($\mathbf{x}$,t) is used for slow variables. The slow time and spatial dependence in $I_k(\mathbf{x}, t)$ corresponds to modulations with a "slow" wavevector $\mathbf{q} \ll \mathbf{k}$. Angle brackets in (A5) stand for ensemble average, which is equivalent to a time average with appropriate ergodic assumptions.

The evolution equation for $I_k(\mathbf{x}, t)$ is derived from (A4) by averaging it over fast scales and by taking the Fourier transform over the slow variable $\mathbf{x}$. Setting $\mathbf{k}' = -\mathbf{k} + \mathbf{q}$

and applying the operator $\int d^2q \exp(i\mathbf{q}\cdot\mathbf{x})$, we obtain

$$\frac{\partial}{\partial t} I_k(\mathbf{x},t) + i\int d^2q \exp(i\mathbf{q}\cdot\mathbf{x})\left(\omega_k + \omega_{-k+q}\right)\left\langle F_k^> F_{-k+q}^> \right\rangle$$

$$+S_1 + S_2 = 0, \tag{A6}$$

$$S_1 = \int\int d^2pd^2q \exp(i\mathbf{q}\cdot\mathbf{x})\left\langle F_{-k+q}^> F_{k-p}^> \right\rangle M_{p,k-p} F_p^<, \tag{A7}$$

$$S_2 = \int\int d^2pd^2q \exp(i\mathbf{q}\cdot\mathbf{x})\left\langle F_{-k+q-p}^> F_k^> \right\rangle M_{p,-k+q-p} F_p^<. \tag{A8}$$

The second term in (A4) gives

$$i\int d^2q \exp(i\mathbf{q}\cdot\mathbf{x})\left(\omega_k + \omega_{-k+q}\right)\left\langle F_k^> F_{-k+q}^> \right\rangle$$

$$= \frac{\partial \omega_k}{\partial \mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{x}} I_k(\mathbf{x},t) - 2\gamma_k I_k, \tag{A9}$$

where γ_k is the linear growth rate, and only the real part of the frequency is presumed for ω_k on the right hand side of this equation.

The ensemble average in S_1 can be transformed by using the inverse of (A5)

$$\left\langle F_{-k+q}^> F_{k-p}^> \right\rangle = \left\langle F_{k-p}^> F_{-(k-p)+q-p}^> \right\rangle = \int d^2x' I_{k-p}(x') \exp(-i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}'). \tag{A10}$$

By using (A10) and expanding in $\mathbf{p} \ll \mathbf{k}$, the expression for S_1 is transformed to

$$S_1 = \int d^2p \exp(i\mathbf{p}\cdot\mathbf{x}) M_{p,k-p}\left(I_k(\mathbf{x}) - \mathbf{p}\cdot\frac{\partial I_k(\mathbf{x})}{\partial \mathbf{k}}\right) F_p^<. \tag{A11}$$

Similarly, by using the identity analogous to (A10) and expanding the interaction coefficient $M_{p,k-p}$ in $\mathbf{p} \ll \mathbf{k}$, we transform S_2 to the form

$$S_2 = \int\int d^2pd^2q \exp(i\mathbf{q}\cdot\mathbf{x})\left(M_{p,-k} + (\mathbf{q}-\mathbf{p})\cdot\frac{\partial M_{p,-k}}{\partial(-\mathbf{k})}\right) F_p^<$$

$$\times \int d^2x' \ \exp(-i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}')I_k(x')$$

$$= \ I_k(x)\int d^2p \exp(i\mathbf{p}\cdot\mathbf{x})M_{p,-k}F_p^< - i\int d^2p \exp(i\mathbf{p}\cdot\mathbf{x})\frac{\partial M_{p,-k}}{\partial(-\mathbf{k})}\cdot\frac{\partial I_k}{\partial \mathbf{x}}F_p^< \ . \quad \text{(A12)}$$

Combining expressions (A11) and (A12) one obtains

$$S_1 + S_2 \ = \ I_k(x)\int d^2p \exp(i\mathbf{p}\cdot\mathbf{x})\left(M_{p,k-p} + M_{p,-k}\right)F_p^< \quad \text{(A13)}$$

$$- \int d^2p \exp(i\mathbf{p}\cdot\mathbf{x})M_{p,k-p}\mathbf{p}\cdot\frac{\partial I_k(\mathbf{x})}{\partial \mathbf{k}}F_p^< \quad \text{(A14)}$$

$$-i\int d^2p \exp(i\mathbf{p}\cdot\mathbf{x})\frac{\partial M_{p,-k}}{\partial(-\mathbf{k})}\cdot\frac{\partial I_k}{\partial \mathbf{x}}F_p^< \ . \quad \text{(A15)}$$

Using (A2) we obtain

$$M_{p,k-p} + M_{p,-k} = \frac{ic}{B_0(1+\psi)}\mathbf{b}\times\mathbf{p}\cdot\mathbf{k}\frac{4\mathbf{k}\cdot\mathbf{p}}{k^2}p_y. \quad \text{(A16)}$$

By making Fourier transformation and using (A16), the first term in (A14) becomes

$$I_k(x)\int\left(M_{p,k-p} + M_{p,-k}\right)d^2p \exp(i\mathbf{p}\cdot\mathbf{x})F_p^< = -\frac{c}{B_0(1+\psi)}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{b}\times\boldsymbol{\nabla}\frac{\partial \overline{F}}{\partial y}\cdot\mathbf{k}\right)\cdot\frac{4\mathbf{k}}{k^2}I_k \quad \text{(A17)}$$

Similarly, the second term in (A14) takes the form

$$-\int \mathbf{p}\cdot\frac{\partial I_k(\mathbf{x})}{\partial \mathbf{k}}M_{p,k-p}d^2p \exp(i\mathbf{p}\cdot\mathbf{x})F_p^< = -\frac{c}{B_0(1+\psi)}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{b}\times\boldsymbol{\nabla}\frac{\partial \overline{F}}{\partial y}\cdot\mathbf{k}\right)\cdot\frac{\partial I_k}{\partial \mathbf{k}} \quad \text{(A18)}$$

and the last term in (A14) transforms to

$$-i\int\frac{\partial M_{p,-k}}{\partial(-\mathbf{k})}\cdot\frac{\partial I_k}{\partial \mathbf{x}}d^2p \exp(i\mathbf{p}\cdot\mathbf{x})F_p^< \ = \frac{c}{B_0(1+\psi)}\frac{\partial}{\partial \mathbf{k}}\left(\mathbf{b}\times\boldsymbol{\nabla}\frac{\partial \overline{F}}{\partial y}\cdot\mathbf{k}\right)\cdot\frac{\partial I_k}{\partial \mathbf{x}}. \quad \text{(A19)}$$

Finally, combining all terms we have

$$\frac{\partial}{\partial t}\left(\left|\widetilde{F}_k\right|^2\right) + \frac{\partial \omega_{rk}}{\partial \mathbf{k}}\cdot\frac{\partial}{\partial \mathbf{x}}\left(\left|\widetilde{F}_k\right|^2\right) - \frac{\partial \omega_{rk}}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{k}}\left(\left|\widetilde{F}_k\right|^2\right)$$

$$+\frac{\partial}{\partial \mathbf{k}}\left(\frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial y}\cdot\mathbf{k}\right)\cdot\frac{\partial}{\partial \mathbf{x}}\left(\left|\widetilde{F}_k\right|^2\right)$$
$$-\frac{\partial}{\partial \mathbf{x}}\left(\frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial y}\cdot\mathbf{k}\right)\cdot\frac{4\mathbf{k}}{k^2}\left|\widetilde{F}_k\right|^2$$
$$-\frac{\partial}{\partial \mathbf{x}}\left(\frac{c}{B_0(1+\psi)}\mathbf{b}\times\nabla\frac{\partial\overline{F}}{\partial y}\cdot\mathbf{k}\right)\cdot\frac{\partial}{\partial \mathbf{k}}\left(\left|\widetilde{F}_k\right|^2\right)=S_k, \qquad \text{(A20)}$$

Multiplying this equation by k^4 and combining the two last terms we obtain the wave-kinetic equation (25).

Acknowledgments. This research was supported by the Natural Sciences and Engineering Research Council of Canada.

References

Balk, A. M., S. V. Nazarenko, and V. E. Zakharov, Nonlocal turbulence of drift waves, *Sov. Phys. JETP, 71(2)*, 249-260, 1990.

Buneman, O., Excitation of field-aligned sound waves by electron streams, *Phys. Rev. Lett., 10*, 285-287, 1963.

Busse, F. H., and A. C. Or, Convection in a rotating cylindrical annulus: Thermal Rossby waves, *J. Fluid Mech. 166*, 173-187, 1986.

Dimant, Ya. S., and R. N. Sudan, Kinetic theory of the Farley-Buneman instability in the *E*-region of the ionosphere, *J. Geophys. Res., 100*, 14,605-14,623, 1995.

Dimant, Y. S., Nonlinearly saturated state of modified two-stream instability in the lower ionosphere, 1999 Spring AGU Meeting, EoS Supplement, April 27, 1999, p. S275, 1999.

Dubrulle, B., and S. Nazarenko, Interaction of turbulence and large-scale vortices in incompressible 2D fluids, *Physica D, 110*, 123-138, 1997.

Farley, D. T., A plasma instability resulting in field-aligned irregularities in the ionosphere, *J. Geophys. Res., 68*, 6083-6093, 1963.

Farley, D. T., W. E. Swartz, D. L. Hysell, and C. Ronchi, High-resolution radar observations of daytime kilometer-scale wave structure in the equatorial electrojet, *J. Geophys. Res., 99*, 299-307, 1994.

Fejer, B. G., and M. C. Kelley, Ionospheric irregularities, *Rev. Geophys., 18*, 401-454, 1980.

Fu, Z. F., L. C. Lee, and J. D. Huba, A quasi-local theory of the E×B instability in the ionosphere, *J. Geophys. Res., 91*, 3263-3269, 1984.

Hamza, A. M., and J.-P. St.-Maurice, A turbulent theoretical framework for the study of current-driven *E* region irregularities at high latitudes. Basic derivation and application to gradient-free situation, *J. Geophys. Res., 98*, 11,587-11,599, 1993a.

Hamza, A. M., and J.-P. St.-Maurice, A self-consistent fully turbulent theory of auroral E region irregularities, *J. Geophys. Res., 98*, 11,601-11,613, 1993b.

Huba, J. D., and L. C. Lee, Short wavelength stabilization of the gradient drift instability due to velocity shear, *Geophys. Res. Lett., 10*, 357-360, 1983.

Janhunen, P., Perpendicular particle simulation of the E region Farley-Buneman instability, *J. Geophys. Res., 99*, 11,461-11,473, 1994.

Kadomtsev, B. B., Plasma Turbulence, Plenum Press, 1964.

Kudeki, E., D. T. Farley, and B. G. Fejer, Long wavelength irregularities in the equatorial electrojet, *Geophys. Res. Lett., 9*, 684, 1982.

Moiseev, S. S., V. G. Pungin, and V. N. Oraevsky, Nonlinear instabilities in plasmas and hydrodynamics, IOP Publishing, Bristol, 2000.

Oppenheim, M., N. Otani, and C. Ronchi, Saturation of the Farley-Buneman instability via nonlinear electron E×B drifts, *J. Geophys. Res., 100*, 17,273-17,286, 1996.

Ronchi, C., R. N. Sudan, and D. T. Farley, Numerical simulations of large-scale plasma turbulence in the daytime equatorial electrojet, *J. Geophys. Res., 9 6*, 21,263-21,279, 1991.

Sahr, J. D., Observation and theory of the radar aurora, Ph.D. thesis, Cornell Univ., Ithaca, N.Y., 1990.

Sahr, J. D., and D. T. Farley, Three wave coupling in the auroral E region, *Ann. Geophys., 13*, 38-44, 1995.

Sahr, J. D., and B. G. Fejer, Auroral electrojet plasma irregularity theory and experiment: A critical review of present understanding and future directions, *J. Geophys. Res., 101*, 26,893-26,909, 1996.

Schlegel, K., and H. Thiemann, Particle-in-cell plasma simulations of the modifi ed two-stream instability, *Ann. Geophys., 12*, 1091-1100, 1994.

Schlegel, K. and J.-P. St-Maurice, Short wavelength gradient-drift waves at high latitudes, *Ann. Geophys., 1*, 259-264, 1983.

Sharma, S. K., and P. K. Kaw, Modification of type II irregularities by finite amplitude type I waves in equatorial electrojet, *J. Geophys. Res., 91*, 12,118-12,120, 1986.

Smolyakov, A. I., and P. H. Diamond, Generalized action invariants for drift wave-zonal flow systems, *Phys. Plasmas, 6*, 4410-4413, 1999.

Smolyakov, A. I., P. H. Diamond, and V. I. Schevchenko, Zonal flow generation by parametric instability in magnetized plasma and geostrophic fluids, *Phys. Plasmas, 7*, 1349-1351, 2000.

St-Maurice, J.-P., and R. S. Kissack, The role played by thermal feedback in heated Farley-Buneman waves at high latitudes, *Ann. Geophys., 18*, 532-546, 2000.

Sudan, R. N., J. Akinrimisi, and D. T. Farley, Generation of small-scale irregularities in the equatorial electrojet, *J. Geophys. Res., 78*, 240-248, 1973.

Sudan, R. N. and M. J. Keskinnen, Theory of strongly turbulent two-dimensional convection of low-pressure plasma, *Phys. Rev. Lett., 38*, 1869-1872, 1977.

Sudan, R. N., Unified theory of type 1 and type 2 irregularities in the equatorial electrojet, *J. Geophys. Res., 88*, 4853-4860, 1983.

Sudan, R. N., A. V. Gruzinov, W. Horton, and N. Kukharkin, Convective turbulence in weakly ionized plasma, *Phys. Report, 283*, 95-119, 1997.

Vedenov, A. A., A. V. Gordeev, and L. I. Rudakov, Oscillation and instability of a weakly turbulent plasma, *Plasma Physics, 9*, 719 - 735, 1967.

Received ____________________