Electromagnetic forces and internal stresses in dielectric media

Winston Frias* and Andrei I. Smolyakov

Department of Physics and Engineering Physics, University of Saskatchewan, 116 Science Place Saskatoon, SK S7N 5E2, Canada

(Received 16 August 2011; revised manuscript received 26 January 2012; published 23 April 2012)

The macroscopic electromagnetic force on dielectric bodies and the related problem of the momentum conservation are discussed. It is argued that different forms of the momentum conservation and, respectively, different forms of the force density, correspond to the different ordering in the macroscopic averaging procedure. Different averaging procedures and averaging length scale assumptions result in expressions for the force density with vastly different force density profiles which can potentially be detected experimentally by measuring the profiles of the internal stresses in the medium. The expressions for the Helmholtz force is generalized for the dissipative case. It is shown that the net (integrated) force on the body in vacuum is the same for Lorentz and Helmholtz expressions in all configuration. The case of a semi-infinite medium is analyzed and it is shown that explicit assumptions on the boundary conditions at infinity remove ambiguity in the force on the semi-infinite dielectric.

DOI: 10.1103/PhysRevE.85.046606 PACS number(s): 03.50.De

I. INTRODUCTION

Electromagnetic forces on charged particles result in an overall force imparted on a body that is called radiation (ponderomotive) pressure. Though these forces are small, their applications are becoming increasingly important (e.g., in manipulations of nanoparticles, optical trapping, and manipulation of biological cells [1–6]). In general, the force is determined by the momentum conservation equation in the medium. Therefore the force exerted by the electromagnetic field on a body is an intrinsic part of the problem of the momentum of light and the form of the electromagnetic stress tensor in a material medium. The two most famous expressions suggested for the momentum of light in the medium are those postulated by Abraham and Minkowski, respectively, given by $\mathbf{g}^A = \mathbf{E}\times\mathbf{H}/c^2$ and $\mathbf{g}^M = \mathbf{D}\times\mathbf{B}$. Different expressions for the momentum result in different expressions for the stress tensor and different expressions for the radiation force on a material medium. There have been numerous expressions for the ponderomotive force published in the literature [7–11] and a number of attempts were made to resolve the problem experimentally [9,12–14]. There also exist a number of review papers that extensively cover the theoretical discussions of this problem as well as experimental results and interpretations [9,12]. In contrast to most of the previous work, we consider one particular aspect that is notably missing in previous discussions. We analyze the spatial distribution of the radiation forces and show how different expressions for macroscopic force correspond to the average of microscopic forces. We argue here that the introduction of the macroscopic force inevitably requires one or another form of averaging and that the various expressions for the radiation forces correspond to different sizes of the sampling volume (averaging length scale). Therefore, different expressions for the forces represent different levels of resolution at which the forces can be measured. Different force densities result in internal stresses existing at different length scales of the averaging volume.

*wpf274@mail.usask.ca

This article is organized as follows. In Sec. II, the problem of the momentum exchange and the force is explored from the microscopic point of view. In Sec. III, the Abraham and Minkowski expressions for the momentum and the stress tensor of the electromagnetic field in a medium are reviewed and discussed from the perspective of the momentum exchange between a different subsystem. In Sec. IV, an expression for the macroscopic force density is derived by using different ordering of the averaging length scales. The force density is compared with the Lorentz force density obtained from the macroscopic Maxwell's equations. In Sec. V, the radiation force on a finite length slab is calculated. In Secs. VI and VII, the force on a dielectric coating film is calculated using two different formulations. In Sec. VIII, the force on a slab separated from a semi-infinite medium by an air gap of finite width is calculated. In Sec. IX, the forces on a finite length dielectric slab with dissipation and on a semi-infinite region are calculated. In Sec. X, the summary and conclusions are given.

II. MOMENTUM EXCHANGE AND FORCES AT THE MICROSCOPIC LEVEL

Conceptually, the exchange of momentum and, respectively, the forces on charged particles are easiest to consider at the microscopic level. The Maxwell equations in microscopic form are

$$\nabla\times\mathbf{b} = \mu_0\mathbf{i} + \frac{1}{c^2}\frac{\partial\mathbf{e}}{\partial t}, \tag{1}$$

$$\nabla\times\mathbf{e} = -\frac{\partial\mathbf{b}}{\partial t}, \tag{2}$$

$$\nabla\cdot\mathbf{e} = \rho/\varepsilon_0, \tag{3}$$

$$\nabla\cdot\mathbf{b} = 0, \tag{4}$$

where ρ and $\mathbf{i}$ are the microscopic charges and currents due to all charged particles and $\mathbf{e}$ and $\mathbf{b}$ are the microscopic electric and magnetic fields. From Eqs. (1)–(4), one can write the conservation of the momentum of the electromagnetic field in the form,

$$\frac{\partial}{\partial t}\,\varepsilon_0\,(\mathbf{e}\times\mathbf{b})+\nabla\cdot\mathbf{t}=-(\rho e+\mathbf{i}\times\mathbf{b}), \tag{5}$$

$$\mathbf{t}=\varepsilon_0\left(\mathbf{ee}-\frac{1}{2}\mathbf{e}^2\mathbf{I}\right)+\frac{1}{\mu_0}\left(\mathbf{b}b-\frac{1}{2}\mathbf{b}^2\mathbf{I}\right). \tag{6}$$

The term on the right-hand side of Eq. (5) represents the momentum exchange between the electromagnetic field and the charged particles. This term, which is simply the total Lorentz force on a charged particle, appears with the opposite sign in the conservation law for the mechanical momentum of particles:

$$\frac{d}{dt}\sum_i m_i\mathbf{v}_i=\mathbf{f}\equiv\rho\mathbf{e}+\mathbf{i}\times\mathbf{b}, \tag{7}$$

where $m\mathbf{v}_i$ is the mechanical momentum of the ith particle and the sum is taken over all particles, $\rho=\sum_i q_i\delta\,(\mathbf{r}-\mathbf{r}_i)$, $\mathbf{i}=\sum_i q_i\mathbf{v}_i\delta\,(\mathbf{r}-\mathbf{r}_i)$, $\mathbf{v}_i=d\mathbf{r}_i/dt$. The microscopic Maxwell stress tensor $\mathbf{t}$ in Eq. (6) describes the flux of the momentum in the electromagnetic field.

Despite its simplicity and transparency, the conservation of the momentum in the form given by Eqs. (5) and (7) is impractical in most cases, except when one is interested in stationary momentum exchange for a finite-size body in vacuum immersed in a harmonic electromagnetic field. In the latter case, the total force on the body is obtained by integration over the whole body (over a closed surface in vacuum just outside the body) and averaging over a time period longer than the wave period, $T>2\pi/\omega$. Then the total stationary force on the body is

$$\mathbf{F}=\int\langle\mathbf{f}\rangle\,dV=-\int dv\left\langle\frac{\partial}{\partial t}\,\varepsilon_0\,(\mathbf{e}\times\mathbf{b})+\nabla\cdot\mathbf{t}\right\rangle$$
$$=-\int_S\mathbf{t}\cdot\mathbf{dS}, \tag{8}$$

where "⟨⟩" means averaging over time. For stationary processes, the average of the time-dependent term is zero and the force becomes equal to the total flux of the momentum into the body (represented by the stress tensor $\mathbf{t}$). The stress tensor is calculated at the surface of the body (on its vacuum side). Therefore it becomes equal to the stress tensor of the macroscopic field in vacuum which can be relatively easily calculated:

$$\mathbf{T}=\varepsilon_0\left(\mathbf{EE}-\frac{1}{2}\mathbf{E}^2\mathbf{I}\right)+\frac{1}{\mu_0}\left(\mathbf{BB}-\frac{1}{2}\mathbf{B}^2\mathbf{I}\right). \tag{9}$$

The problem of averaging (length and time scales) is already apparent in the steps involved in Eq. (8). Averaging over the whole body volume and taking the boundary of the integration volume just outside of the body allows one to replace the microscopic fields with the macroscopic ones, which can be easily justified in vacuum. In this case $\mathbf{t}\to\mathbf{T}$ and the surface values for the macroscopic field and stress tensor $\mathbf{T}$ can easily be determined by solving the macroscopic scattering problem for a harmonic electromagnetic field incident on the body. The full time derivative term will average to zero only for periodic processes when the time average is done over a time scale longer than the wave period, $T>2\pi/\omega$. One has to assume also that the local instantaneous fields $\mathbf{e}$ and $\mathbf{b}$ do not involve correlated fast scale processes in which quadratic averages may contribute to the slow scale evolution of the fields.

III. ABRAHAM AND MINKOWSKI FORMS OF THE MOMENTUM CONSERVATION AND THE LORENTZ FORCE FROM MACROSCOPIC EQUATIONS

The expressions for the momentum and forces at the microscopic level, Eqs. (5) and (6), illustrate the momentum conservation between two subsystems: The momentum lost by the electromagnetic field appears as the particle momentum. This momentum exchange represents the force on all particles. In the microscopic equations both subsystems are well defined and the momentum exchange and force can be clearly identified. The situation becomes less transparent for macroscopic Maxwell's equations.

Let's consider macroscopic Maxwell's equations in the form:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{10}$$

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}, \tag{11}$$

$$\nabla\cdot\mathbf{D}=0, \tag{12}$$

$$\nabla\cdot\mathbf{B}=0. \tag{13}$$

Assuming ideal dielectric and neglecting free charges and currents, by manipulating Eqs. (10)–(13) in the same way as in Eqs. (1)–(4), one can construct a conservation equation of the form,

$$\frac{\partial\mathbf{g}^M}{\partial t}+\nabla\cdot\mathbf{T}^M=-\mathbf{f}^H, \tag{14}$$

where

$$\mathbf{g}^M=\mathbf{D}\times\mathbf{B} \tag{15}$$

is the Minkowski momentum [15],

$$T^M_{i,j}=\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D}\,\delta_{ij})-E_i\,D_j+\tfrac{1}{2}(\mathbf{H}\cdot\mathbf{B}\,\delta_{ij})-H_i\,B_j \tag{16}$$

is the Minkowski stress tensor, and

$$\mathbf{f}^H=-\tfrac{1}{2}\,E^2\,\nabla\varepsilon-\tfrac{1}{2}\,H^2\,\nabla\mu \tag{17}$$

is the exchange term.

Equation (14) has the structure of the conservation of the density of the Minkowski momentum, the term $\nabla\cdot\mathbf{T}^M$ describes the flux of the momentum, and the exchange term, which is known as the Helmholtz force, is given by $\mathbf{f}^H$.

Further identical manipulations with Maxwell's equation result in the momentum conservation in the Abraham form [16]:

$$\frac{\partial\mathbf{g}^A}{\partial t}+\nabla\cdot\mathbf{T}^A=-\mathbf{f}^A, \tag{18}$$

where

$$\mathbf{g}^A=\mathbf{E}\times\mathbf{H}/c^2 \tag{19}$$

is the Abraham momentum, and

$$T^A_{i,j} = \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D}\,\delta_{ij} - E_i\,D_j - E_j\,D_i) + \tfrac{1}{2}(\mathbf{H}\cdot\mathbf{B}\,\delta_{ij} - H_i\,B_j - H_j\,B_i) \tag{20}$$

is the Abraham stress tensor. In this formulation, the exchange term is given by

$$\mathbf{f}^A = -\frac{1}{2}\,E^2\,\nabla\varepsilon - \frac{1}{2}\,H^2\,\nabla\mu + \frac{\varepsilon\mu - 1}{c^2}\,\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t}, \tag{21}$$

which is different from the Helmholtz force in Eq. (17) by the addition of the last term, the so-called Abraham force [16]. It is important to note at this point that both formulations of the momentum conservation (14) and (18) are mathematically equivalent and both must be correct, from a mathematical point of view, as long as we accept as correct the macroscopic Maxwell equations (10)–(13).

Another and very attractive formulation of the momentum conservation, with transparent physical meaning, can be obtained from the macroscopic Maxwell equations with explicit source terms due to the polarization charges and the magnetization current in the absence of free charges and currents:

$$\rho_m = -\nabla\cdot\mathbf{P}, \tag{22}$$

$$\mathbf{J}_m = \nabla\times\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t}. \tag{23}$$

The source terms in Eqs. (22) and (23) create macroscopic electric and magnetic field according to Maxwell equations,

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{24}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}_m + \mu_0\,\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{25}$$

$$\nabla\cdot\mathbf{E} = \rho_m/\varepsilon_0, \tag{26}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{27}$$

Straightforward identical manipulations with Eqs. (22)–(27) produce the following form of the conservation law:

$$\frac{\partial\mathbf{g}}{\partial t} + \nabla\cdot\mathbf{T} = -\mathbf{f}^L, \tag{28}$$

where

$$\mathbf{g} = \varepsilon_0\,\mathbf{E}\times\mathbf{B} \tag{29}$$

is the electromagnetic field momentum,

$$T_{i,j} = \frac{1}{2}\varepsilon_0\,\mathbf{E}\cdot\mathbf{E}\,\delta_{ij} - \varepsilon_0\,E_i\,E_j + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\,\delta_{ij} - \frac{1}{\mu_0}\,B_i\,B_j \tag{30}$$

is the flux of the field momentum, and

$$\mathbf{f}^L = \rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B} \tag{31}$$

is the Lorentz force acting on the polarization charges and the magnetization and polarization currents. It is important to note that **E** and **B** are the macroscopic fields calculated *inside* the medium and that both the electromagnetic tensor and the momentum density have the same form in both vacuum and medium. Of course the fields themselves, **E** and **B**, are different in vacuum and medium. In the latter, the fields are created by the polarization charges and magnetization currents ρ_m and $\mathbf{J}_m$, while in vacuum, **E** and **B** are the source free vacuum field determined by the Maxwell equations and boundary conditions.

The momentum conservation in the form given in Eq. (28) has an attractive feature of clearly showing the momentum exchange between the macroscopic electromagnetic field in the medium and the medium itself (via the polarization charge and the polarization and magnetization currents). The time variation of the momentum density and flux in Eq. (28) refer to the electromagnetic field part only. The exchange term $\mathbf{f}^L$ has the simple interpretation of the total force acting on the polarization and magnetization charges and currents due to the macroscopic fields. Equation (28) represents a momentum balance in a clearly defined subsystem of the macroscopic electromagnetic field. This momentum balance is in the proper form of the conservation law (e.g, density of the momentum in a given volume, flux of the momentum through the boundary, and sink/source terms). The sink/source which represents momentum lost by the fields is easily identifiable with the force applied to the other subsystem, the material medium. This is the total electromagnetic force applied to the polarization charges and magnetization currents.

We should note that contrary to the Lorentz force formulation (28), both the Abraham and Minkowski formulations of the momentum conservation do not show clear separation of two different subsystems with a momentum exchange between them. Obviously, both $\mathbf{g}^A$ and $\mathbf{g}^M$, as well as $\mathbf{T}^A$ and $\mathbf{T}^M$, contain a mixture of electromagnetic field terms **E** and **B**, and material terms **P** and **M** (or **D** and **H**) [17]. Respectively, the momentum exchange terms for both the Abraham and Minkowski expressions (sink/source terms) are not easily identifiable with the momentum exchange between two different subsystems. Note that for Lorentz force the momentum density and momentum flux terms in Eq. (28) contain only the electromagnetic field terms [and not a mixture of fields and material terms as in Eqs. (14) and (18).

A notable feature of the Abraham and Helmholtz (Minkowski) forces, $\mathbf{f}^A$ and $\mathbf{f}^H$, is that in the stationary case, when the time-dependent Abraham force can be neglected, the forces occur at the interfaces of the inhomogeneous material medium. The Abraham and Helmholtz (Minkowski) force densities in the stationary case are zero inside the body even in those cases when the electromagnetic field amplitude is inhomogeneous. For a finite slab of a homogeneous dielectric subject to an incident plane electromagnetic field, the force density will be a sum of two delta-function-like contributions at the front and back ends of the slab. Only the field amplitudes at the interfaces enter the expressions for the force. Of course, the field amplitudes at the boundaries (and hence the total force) implicitly depend on the internal distribution of the electromagnetic field.

One can notice that formal transformations between the Abraham and Minkowski conservation forms involve the transformation between the volume and surface contributions. Therefore, after volume averaging (volume integration) certain volume contribution will appears only in the form of surface terms. This situation is fully equivalent to the usual pressure force used in fluid dynamics. The volume (bulk) gradient pressure force describes the internal pressure force. After averaging over a given volume, the total force is simply given by the surface contributions.

In summary of this section, we would like to emphasize again that as long as one accepts the validity of the macroscopic Maxwell equations in the form (10)–(13) and (1)–(4), all three forms of the conservation law are mathematically correct since they were obtained by formal mathematical transformations of Maxwell's equations. The important question is how the exchange terms in Eqs. (14), (18), and (28), given by Eqs. (17), (21), and (31) should be interpreted, how these exchange terms are related to some measurable physical forces, and where these forces are applied (what is the spatial distribution of the force density).

In subsequent sections we would like to argue that the spatial part of the force density in the Minkowski and Abraham expressions, which are purely surface force densities in a homogeneous dielectric, suggest an averaging size of the order of the dimensions of the dielectric body. As a result, any internal stresses are averaged out and reduced to the remaining surface contributions [18]. We show also that the force density given by the Lorentz force acting on the induced charge and current densities $\rho = -\nabla \cdot \mathbf{P}$ and $\mathbf{J} = \partial \mathbf{P}/\partial t + \nabla \times \mathbf{M}$, is a volume force and suggests an averaging sample of the scale at which the induced macroscopic charges and currents vary more slowly than the fields in the volume of interest. The latter corresponds to neglecting the internal fields created by the individual charges and currents inside certain macroscopic averaging volume.

IV. THE AVERAGING OF THE MICROSCOPIC FORCE

In this section, the macroscopic force is obtained by averaging the microscopic forces acting on the point charges comprising the medium. The total force is defined as the sum of the forces on all particles inside a given volume. This summation (averaging) eliminates the internal forces acting between the particles inside the averaging volume sample. Obviously, changing the size of the averaging sample changes the expression for the force density. We review two standard derivations of the average force [17,19] and consider the physical conditions under which one or another approach is applicable.

Consider a volume of size a, where a is much smaller than the physical size of the medium but much larger than the distance between microscopic charges, so there are several charges present inside the volume. For a dielectric, in the absence of free charges, the volume a involves one neutral atom (or a small group of atoms clustered in the molecule). The total force on this volume element is

$$\mathbf{F}_l = \sum_a q_i \left(\mathcal{E}\left(\mathbf{r}_i\right) + \mathbf{v}_i \times \mathcal{B}\left(\mathbf{r}_i\right)\right), \tag{32}$$

where the sum is taken over all charged particles inside the volume a. This volume is labeled by the index l, so $\mathbf{F}_l$ is the total force acting on the volume a. One can also write Eq. (32) in the form,

$$\mathbf{F} = \int_a \rho\mathcal{E} + \mathbf{J} \times \mathcal{B}\, dV, \tag{33}$$

where ρ and $\mathbf{J}$ are the charge and current densities inside a, which are in general, rapidly changing functions of the spatial coordinates. The electric and magnetic fields, $\mathcal{E}$ and $\mathcal{B}$, inside the volume a are produced by the sources *internal* and *external* to the volume. The part of the force produced by the fields of the internal sources (which change very rapidly inside a) will cancel out due to Newton's third law, and the net force will be then produced only by the fields due to the *external* (to the volume) sources, $\mathbf{E}$ and $\mathbf{B}$. We set the size a, such that the external fields can be expanded around the center of the volume as

$$\mathbf{E}(\mathbf{r}) \approx \mathbf{E}(\mathbf{r_0}) + (\mathbf{r} - \mathbf{r_0}) \cdot \nabla\mathbf{E}(\mathbf{r_0}), \tag{34}$$

$$\mathbf{B}(\mathbf{r}) \approx \mathbf{B}(\mathbf{r_0}) + (\mathbf{r} - \mathbf{r_0}) \cdot \nabla\mathbf{B}(\mathbf{r_0}). \tag{35}$$

The latter expansion requires the condition $a < L$, where $L \simeq (E^{-1}\partial E/\partial r) \simeq (B^{-1}\partial B/\partial r)$. Assuming the total charge inside the volume to be zero, one gets the expression for the force acting on a point dipole in an inhomogeneous electromagnetic field [10,17,19,20]:

$$\begin{aligned}\mathbf{F}_l &= \int_a \{\rho_l\, \mathbf{r} \cdot \nabla\mathbf{E}(0) + \mathbf{J}_l \times (\mathbf{B(0)} + \mathbf{r} \cdot \nabla\mathbf{B}(0))\}\, dV \\ &= (\mathbf{p}_l \cdot \nabla)\mathbf{E}(0) + \int_a \mathbf{J}_l \times \mathbf{B}(0)\, dV \\ &\quad + \int_a \mathbf{J}_l \times \mathbf{r} \cdot \nabla\mathbf{B}(0)\, dV.\end{aligned} \tag{36}$$

In Eq. (36), the definition of the dipole moment $\mathbf{p}_l = \int_a \rho_l\, \mathbf{r}\, dV$ was used.

One can further convert the two integrals in the right-hand side of Eq. (36). Using the continuity equation $\partial\rho_l/\partial t = -\nabla \cdot \mathbf{J}_l$, the first integral can be expressed as

$$\begin{aligned}F_{1i} &= \varepsilon_{ijk}\left\{B_k(0) \int_a \nabla \cdot (\mathbf{J}_l\mathbf{r})_j\, dV + B_k(0) \int_a \frac{\partial\rho_l}{\partial t}\, \mathbf{r}_j\, dV\right\} \\ &= \varepsilon_{ijk}\, \frac{\partial p_{l_j}}{\partial t}\, B_k(0), \\ \mathbf{F}_1 &= \frac{\partial\mathbf{p}_l}{\partial t} \times \mathbf{B}.\end{aligned} \tag{37}$$

For the second integral, using [19]

$$\mathbf{x} \cdot \int_a \mathbf{r}\, J_i\, dV = -\frac{1}{2}\left\{\mathbf{x} \times \int_a \mathbf{r} \times \mathbf{J}\, dV\right\},$$

we have

$$\begin{aligned}F_{2i} &= \varepsilon_{ijk}\frac{1}{2}\left\{\int_a (\mathbf{r} \times \mathbf{J})\, dV \times \nabla\right\}_j B_k(0) \\ &= \varepsilon_{ijk}(\mathbf{m} \times \nabla)_j\, B_k(0), \\ \mathbf{F}_2 &= (\mathbf{m} \times \nabla) \times \mathbf{B},\end{aligned} \tag{38}$$

where $\mathbf{m}$ is the magnetic moment of the dipole. Combining the results from Eqs. (36)–(38), the total force on the lth dipole is given by

$$\mathbf{F}_l = (\mathbf{p}_l \cdot \nabla)\mathbf{E} + \frac{\partial\mathbf{p}_l}{\partial t} \times \mathbf{B} + (\mathbf{m}_l \times \nabla) \times \mathbf{B}. \tag{39}$$

It is important to note that in this derivation $\mathbf{p}_l$ and $\mathbf{m}_l$ are constants and the electromagnetic field is assumed to be slowly varying on the length scale a. Now, one can group individual dipole elements into a larger sampling size L_m, $a < L_m < L$. Then the total force density on the group (cluster) of dipoles is obtained by summation over all dipoles in the cluster. By dividing the net force by the volume one gets the force density,

$$\mathbf{f}^d = (1/V)\sum_l \mathbf{F}_l = (\mathbf{P}\cdot\nabla)\mathbf{E} + \frac{\partial \mathbf{P}}{\partial t} \times \mathbf{B} + (\mathbf{M}\times\nabla)\mathbf{B}. \quad (40)$$

In this last equation, the density of polarization and magnetization vectors $\mathbf{P}$ and $\mathbf{M}$ are defined as [21]

$$\mathbf{P} = (1/V)\sum_l \mathbf{p}_l, \quad \mathbf{M} = (1/V)\sum_l \mathbf{m}_l.$$

Note that in the transformation from Eqs. (39) to (40) the electromagnetic field is assumed to be slowly varying on the length scale L_m (strictly speaking one requires that the gradients of the field $\nabla\mathbf{E}$ and $\nabla\mathbf{B}$ are approximately uniform on the length scale L_m). Therefore, the dipole force in Eq. (40) is a result of the averaging over length scale L_m which involves many individual dipoles with approximately uniform gradients of the electromagnetic field over L_m.

It is worth noting that Eq. (40) is valid for dispersive media, provided that the dispersion of $\mathbf{P}(\mathbf{E},\omega)$ and $\mathbf{M}(\mathbf{B},\omega)$ are defined. In particular, for a plasma, where $\mathbf{P} = -\omega_{pe}^2\mathbf{E}/\omega^2$, the force density in Eq. (40) can be reduced to the familiar expression for the ponderomotive or Miller force, proportional to the gradient of E^2,

$$\langle \mathbf{f}^d \rangle = -\frac{\omega_p^2}{\omega^2}\nabla\frac{|E|^2}{4}.$$

A conservation equation can be constructed that involves the dipole force $\mathbf{f}^d$ similar to Eq. (28):

$$\frac{\partial \mathbf{g}}{\partial t} + \nabla\cdot\mathbf{T}^d = -\mathbf{f}^d, \quad (41)$$

where

$$\mathbf{g} = \varepsilon_0\,\mathbf{E}\times\mathbf{B}, \quad (42)$$

$$T^d_{i,j} = \frac{\varepsilon_0}{2}\mathbf{E}\cdot\mathbf{E}\,\delta_{ij} - \varepsilon_0\,E_i\,E_j - P_i\,E_j + \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}\delta_{ij} - \mathbf{M}\cdot\mathbf{B}\,\delta_{ij} - \frac{B_i\,B_j}{\mu_0} + M_i B_j. \quad (43)$$

There exists a different ordering of the length scales leading to a different expression for the macroscopic force. In this common approach [17,22], an opposite assumption is made with respect to the relative length scale of the charge and current distribution and the electric field. In this approach the averaging is done over the volume size $a \ll L$ by using some weight function w [17,19], such that

$$\mathbf{f} = \left(\int_L \rho(\mathbf{r})\,w(\mathbf{r}-\mathbf{s})\,d^3s\right)\mathbf{E}(\mathbf{r}) + \left(\int_L \mathbf{J}(\mathbf{r})\,w(\mathbf{r}-\mathbf{s})\,d^3s\right)\times\mathbf{B}(\mathbf{r}), \quad (44)$$

The volume size a involves many individual charges which are summed up, but the electric and magnetic fields are assumed to be uniform over the scale length a. The weight function w is expanded in a Taylor series [17] to take into account weak variations of charge distribution leading to a finite electric charge within the averaging volume (polarization charge density),

$$w(\mathbf{r}+\mathbf{d}) = w(\mathbf{r}) + (\mathbf{d}\cdot\nabla)w(\mathbf{r}). \quad (45)$$

Then, the integrals inside the brackets in Eq. (44) yield the macroscopic Lorentz force density which was obtained in the previous section, Eq. (31) [17],

$$\mathbf{f}^L = (-\nabla\cdot\mathbf{P})\,\mathbf{E} + \left(\frac{\partial\mathbf{P}}{\partial t} + \nabla\times\mathbf{M}\right)\times\mathbf{B}. \quad (46)$$

The two expressions in Eqs. (40) and ((46)) are related as

$$\mathbf{f}^d = \mathbf{f}^L + \nabla\cdot(\mathbf{PE} - \mathbf{BM} + \mathbf{B}\cdot\mathbf{M}\,\mathbf{I}). \quad (47)$$

Furthermore, the dipole force is related to the Helmholtz force via

$$\mathbf{f}^d = \mathbf{f}^H + \nabla\cdot\left(\tfrac{1}{2}\mathbf{P}\cdot\mathbf{E}\,\mathbf{I} + \tfrac{1}{2}\mathbf{M}\cdot\mathbf{B}\,\mathbf{I}\right). \quad (48)$$

From Eqs. (47) and (48), it can be seen that the force densities differ by the full derivative of a tensor. This tensor should be interpreted as an internal pressure (stress) tensor that can, in principle, be measured. As can be seen from Eqs. (47) and (48), upon integration over the whole volume of the body, the internal pressure terms will vanish when the surface of the integration volume is extended into vacuum where polarization and magnetization are zero and all the expressions in Eqs. (14), (18), (28), and (41) are equivalent. In the case of nonmagnetic media, when $\mathbf{M} = \mathbf{B} = \mathbf{0}$, and for transverse waves when $\nabla\cdot(\mathbf{PE}) = 0$, the dipole and the Lorentz force are equal, as can be seen from Eq. (47). For nonmagnetic media, the Helmholtz force differs from both the dipole and the Lorentz forces by a factor $\frac{1}{2}\nabla(\mathbf{P}\cdot\mathbf{E})$.

V. PONDEROMOTIVE FORCE DENSITY IN A DIELECTRIC SLAB OF THE FINITE LENGTH

As it was shown in the previous sections, in general, for a lossless medium, the different formulations for the ponderomotive force may have a surface contribution as in Eqs. (17) and (21), or a volume contribution as in Eqs. (40) and (46). In the ideal case (absence of the dissipation) the surface part is represented by delta functions localized at the interfaces, where the dielectric index is discontinuous. In this section, we will calculate and compare the force profiles for a nondissipative slab of a finite and fixed length d.

We are concerned only with the interaction of monochromatic high-frequency electromagnetic fields with homogeneous dielectric, nonmagnetic media. In this case, the time average quantities can be replaced with their respective averages over a wave cycle. This way, the full time derivatives of the momentum density terms $\partial_t\mathbf{g}$ and similar term in the force (such as Abraham force) will average to zero. Furthermore, the conservation equations will reduce to the form,

$$\nabla\cdot\langle\mathbf{T}\rangle = -\langle\mathbf{f}\rangle. \quad (49)$$

The total force on the body can be calculated by integrating the force density over the whole volume of the body (or equivalently, integrating the stress tensor over the surface of the body) as in Eq. (8),

$$\langle \mathbf{F} \rangle = -\int_V \langle \mathbf{f} \rangle \, dV \, dV = \oint_s \langle \mathbf{T} \rangle \cdot d\mathbf{S}. \tag{50}$$

Consider a nonmagnetic dielectric slab of thickness d. The slab is characterized by a dielectric constant ε and bounded by vacuum (air) on both sides. Taking the z axis as normal to the slab and linearly polarized monochromatic light normally incident on the slab, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are given by

$$\mathbf{E} = [E_x, 0, 0] e^{-i\omega t}, \tag{51}$$

$$\mathbf{B} = \left[0, -\frac{i}{\omega}\frac{\partial E_x}{\partial z}, 0\right] e^{-i\omega t}. \tag{52}$$

For the transverse waves in Eqs. (51) and (52), the electric contribution from the dipole force $(\mathbf{P}\cdot\nabla)\mathbf{E}$ and the Lorentz force $(-\nabla\cdot\mathbf{P})\mathbf{E}$ are zero. As a result, the dipole and Lorentz force densities are the same and given by the force density acting on the polarization current [10],

$$\langle \mathbf{f}^d \rangle = \langle \mathbf{f}^L \rangle = \frac{1}{2}\mathrm{Re}\left(\frac{\partial \mathbf{P}}{\partial t} \times \mathbf{B}^*\right), \tag{53}$$

where $\mathbf{P} = \varepsilon_0(\varepsilon - 1)\,\mathbf{E}$. In what follows, we refer to both only as Lorentz force density/force.

For a dielectric slab, the electric field is given by

$$E_x = (e^{ik_v z} + \mathrm{R}e^{-ik_v z})E_0 \quad z > 0,$$
$$E_x = (\mathrm{A}e^{ik_1 z} + \mathrm{B}e^{-ik_1 z})E_0 \quad 0 < z < d,$$
$$E_x = \mathrm{T}e^{ik_v z}E_0 \quad d < z,$$

where $k_v = \omega/c$ and $k_1 = k_v\sqrt{\varepsilon}$. The average force density Eq. (53) becomes

$$\langle \mathbf{f}^L \rangle = \hat{\mathbf{z}}\varepsilon_0(\varepsilon - 1)k_1|E_0|^2 \mathrm{Im}(A^*\mathrm{B}e^{-2ik_1 z}). \tag{54}$$

The coefficients in Eq. (54) are obtained from the solution of the scattering problem in the form,

$$\mathrm{Im}\{A^*\mathrm{B}e^{-2ik_1 z}\} = \frac{(\varepsilon - 1)\sin(2k_1(d - z))}{4\varepsilon\cos^2(k_1 d) + (\varepsilon + 1)^2\sin^2(k_1 d)},$$

such that the time-averaged force density can be written as

$$\langle \mathbf{f}^L \rangle = \hat{\mathbf{z}}|E_0|^2 \frac{\varepsilon_0 k_1(\varepsilon - 1)^2\sin(2k_1(d - z))}{4\varepsilon\cos^2(k_1 d) + (\varepsilon + 1)^2\sin^2(k_1 d)}. \tag{55}$$

The total force is obtained by the integration over the slab length of Eq. (55),

$$\langle \mathbf{F}^L \rangle = \hat{\mathbf{z}}S\int_0^d |E_0|^2 \frac{\varepsilon_0 k_1(\varepsilon - 1)^2\sin(2k_1(d - z))}{4\varepsilon\cos^2(k_1 d) + (\varepsilon + 1)^2\sin^2(k_1 d)}\,dz, \tag{56}$$

with S being the cross-sectional area of the slab. The expression for the total force reduces to

$$\begin{aligned}\langle F^L \rangle &= \frac{S\varepsilon_0(\varepsilon - 1)^2|E_0|^2\sin^2(k_1 d)}{4\varepsilon\cos^2(k_1 d) + (\varepsilon + 1)^2\sin^2(k_1 d)} \\ &= \frac{1}{2}S\varepsilon_0|E_0|^2|(1 + |R|^2 - |T|^2) \\ &= S\varepsilon_0|E_0|^2|R|^2,\end{aligned} \tag{57}$$

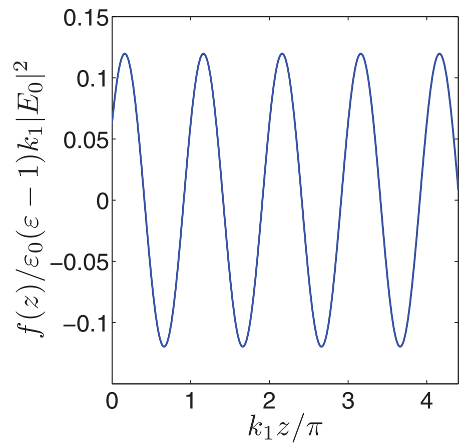


FIG. 1. (Color online) Density of the Lorentz force in a dielectric slab, as calculated from Eq. (55).

where R and T are the reflection and transmission coefficients.

As can be seen from Fig. 1, the Lorentz force density has a standing wave spatial profile inside the slab, being positive or negative according to the sign of $\sin(2k_1(d - z))$. The total force oscillates as a function of the slab length and dielectric constant according to oscillations of the reflection coefficient as shown in Figs. 2(a) and 2(b). Because the total force is proportional to $|R|^2$, it is always positive (pushes the slab), despite the fact that the force density oscillates between positive and negative values. The total force is zero when when the reflectance is zero, $R = 0$.

The Helmholtz force density, being proportional to $\nabla\varepsilon$, is a purely surface force density given by the sum of two Dirac delta functions, one at each interface:

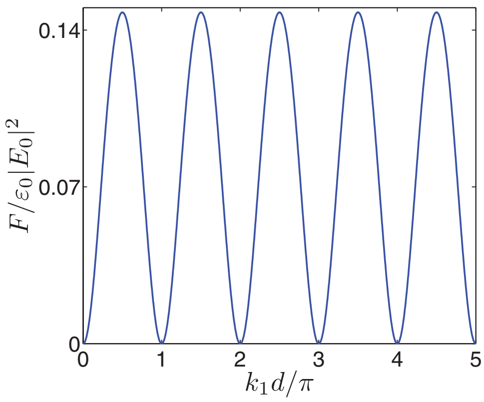
$$\langle \mathbf{f}^H \rangle = -\hat{\mathbf{z}}\tfrac{1}{4}|E_x|^2\,\varepsilon_0(\varepsilon - 1)\,\{\delta(z) - \delta(z - d)\}. \tag{58}$$

Direct integration of Eq. (58) over the width of the slab yields the total Helmholtz force,

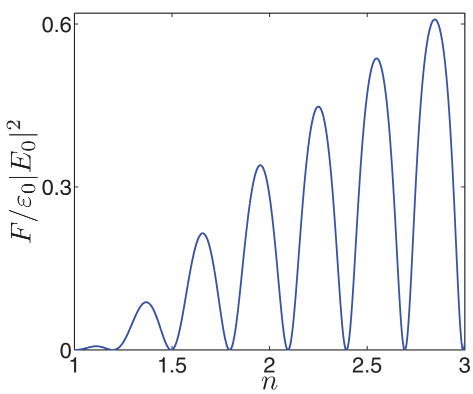
$$\begin{aligned}\langle \mathbf{F}^H \rangle &= -\hat{\mathbf{z}}\frac{1}{4}S\int_0^d |E_x|^2\,\varepsilon_0(\varepsilon - 1)\,\{\delta(z) - \delta(z - d)\}\,dz \\ &= \hat{\mathbf{z}}\frac{1}{2}S\varepsilon_0|E_0|^2|(1 + |R|^2 - |T|^2) \\ &= \hat{\mathbf{z}}S\varepsilon_0|E_0|^2|R|^2.\end{aligned} \tag{59}$$

As it follows from Eqs. (57) and (59), the total Lorentz force is equal to the total Helmholtz force. This is the general result, which simply follows from the relations (47) and (48). The force densities differ by a full divergence of the internal stress tensor, the contribution of which vanishes upon integration over the total slab volume bounded by vacuum, since in vacuum $\mathbf{P} = 0$ and $\mathbf{M} = \mathbf{0}$.

The general results in Eqs. (57) and (59) also follow from a simple balance of the exchange of the momentum between photons and the dielectric slab. Consider a photon beam normally incident on the dielectric layer. The incident beam has a photon flux $N_i\,c$, where N_i is a photon density $N_i = w/(\hbar\omega)$, time-average energy density $w = \varepsilon_0|E_0|^2/2$, and momentum flux $\tau_i = N_i\hbar k$. The reflected beam will have a momentum $\tau_r = N_r\hbar k$, where $N_r = |R|^2\,N_i$. The transmitted beam with a flux N_t has a momentum flux $\tau_t = N_t\hbar k$ with $N_t = |T|^2\,N_i$.



(a)



(b)

FIG. 2. (Color online) Total Lorentz force on the dielectric slab as calculated using Eqs. (57) and (59): (a) as a function of the width, $n = \sqrt{\varepsilon} = 1.5$; (b) as a function of n.

The total force per unit area applied to the dielectric is then

$$F = \hbar k\,(N_i + N_r - N_t) = \hbar k\, N_i\,(1 + |R|^2 - |T|^2), \qquad (60)$$

and obviously F is positive for any $|T|^2 < 1$ or since $|R|^2 + |T|^2 = 1$, the force becomes $F = 2\hbar k\, N_i\,|R|^2 = \varepsilon_0 |E_0|^2 |R|^2$.

In summary, we would like to point out that while the total Lorentz and Helmholtz forces are the same, the corresponding force densities have very different spatial profiles: The Helmholtz force density is purely a surface force density, and the Lorentz force density has an oscillating density profile, as can be seen in Fig. 1.

VI. LORENTZ FORCE ON A DIELECTRIC COATING FILM

In this section, we consider the forces on a layer of a dielectric film coated on a material with different dielectric constant. This example provides a good insight on the nature of the ponderomotive forces acting on a medium and emphasize the differences between Lorentz and Helmholtz forces. These results are also relevant to recent experiments measuring the force on the ends of the dielectric fibers [13].

We consider a dielectric film coating of permittivity ε_1 and thickness d on a semi-infinite substrate with dielectric constant ε_2. The electromagnetic wave is incident from the vacuum region on the left. For such a configuration, the electric field

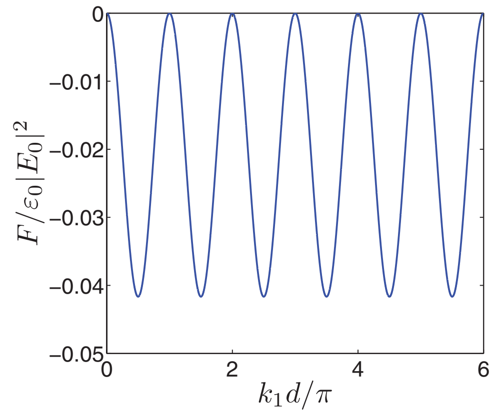


FIG. 3. (Color online) The total force on the coating film as calculated from Eq. (62); $\varepsilon_1 = 1.5$ and $\varepsilon_2 = 2.25$.

is given by

$$\begin{aligned} E_x &= (e^{ik_v z} + \mathrm{Re}^{-ik_v z})E_0 \quad z > 0,\\ E_x &= (\mathrm{Ae}^{ik_1 z} + \mathrm{Be}^{-ik_1 z})E_0 \quad 0 < z < d, \\ E_x &= \mathrm{Te}^{ik_2 z}E_0 \quad d < z, \end{aligned} \qquad (61)$$

where $k_v = \frac{\omega}{c}$, $k_1 = k_v\sqrt{\varepsilon_1}$, and $k_2 = k_v\sqrt{\varepsilon_2}$. As in the case we considered in the previous section, the electric components of the Lorentz and dipole force densities do not make any contribution. In this case, the Lorentz and dipole force densities are equal and the force density is given by the force density on the polarization current as in Eq. (53).

The total force is obtained by the integration the force density, given again by Eq. (54), over the layer width, from $z = 0$ to $z = d$:

$$\langle \mathbf{F}^L\rangle = \hat{\mathbf{z}} S\varepsilon_0|E_0|^2 \frac{(\varepsilon_1 - 1)(\varepsilon_1 - \varepsilon_2)}{|g|^2}\sin^2(k_1 d), \qquad (62)$$

where the factor g given by

$$g = \sqrt{\varepsilon_1}(1 + \sqrt{\varepsilon_2})\cos(k_1 d) - i(\varepsilon_1 + \sqrt{\varepsilon_2})\sin(k_1 d).$$

The sign of the force is determined by the sign of the product $(\varepsilon_1 - 1)(\varepsilon_1 - \varepsilon_2)$. After some algebra, it is not difficult to show that the result in Eq. (62) can be written in the form,

$$\langle \mathbf{F}^L\rangle = \hat{\mathbf{z}}\frac{S\varepsilon_0|E_0|^2}{2}\left(1 + |R|^2 - \frac{1 + \varepsilon_2}{2}|T|^2\right). \qquad (63)$$

The force acting on the coating as calculated in Eq. (62) is shown in Fig. 3 as a function of the phase $k_1 d$ inside the dielectric coating film d. The force is zero for $k_1 d = n\pi$, that corresponds to the $\lambda/2$ layer [also compare this with Eq. (57), where the force is zero for $k_1 d = n\pi$].

Note that the force on the $\lambda/4$ antireflective coating is not zero. Applying the result from Eq. (63) to a $\lambda/4$ antireflective layer, such that $d = \lambda_0/4\sqrt{\varepsilon_1}$, $k_1 d = \pi/2$, $\varepsilon_1 = \sqrt{\varepsilon_2}$, we have $|R|^2 = 0$ and $|T|^2 = 1/\sqrt{\varepsilon_2}$. This reduces Eq. (63) to

$$\langle F^L\rangle = -\frac{1}{4}\,\varepsilon_0\, S\,\frac{(\sqrt{\varepsilon_2} - 1)^2}{\sqrt{\varepsilon_2}}\,|E_0|^2. \qquad (64)$$

This expression shows that the Lorentz force on the $\lambda/4$ coating is always negative, so that it pulls the coating from the substrate.

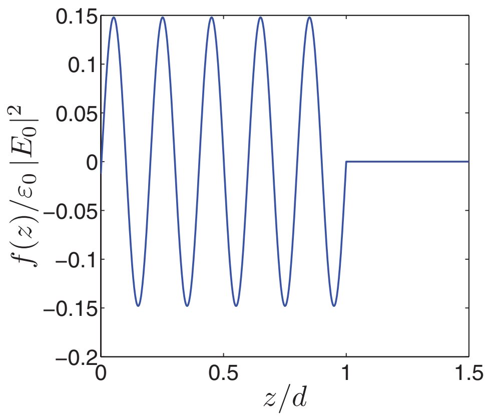


FIG. 4. (Color online) Force density in a coating-substrate system from Eq. (54). The force density is zero inside the substrate ($z > d$); $\varepsilon_1 = 1.5$ and $\varepsilon_2 = 2.25$.

The force density in the coating-substrate system is shown in Fig. 4. One important feature to note from Fig. 4 is that the Lorentz force density is zero (in absence of dissipation) inside the semi-infinite substrate, for $z > d$. Alternatively, this can be illustrated by the running integral of the force density defined by the expression,

$$\langle \mathbf{F}^L(z)\rangle = \hat{\mathbf{z}} S \int_0^z f(z')dz'. \quad (65)$$

The total force is accumulated in the coating layer only. For $z > d$, the total integrated force in Eq. (65) remains constant [only the coating makes a contribution to the integral in Eq. (65)]. Therefore, the force density is zero in the semi-infinite region, which superficially may lead to a certain paradox. It is obvious, however, that a single monochromatic wave, as in the region $z > d$, would not create any force in the semi-infinite region without dissipation. The case of a semi-infinite dielectric region requires a special treatment and it is discussed in Sec. IX.

VII. HELMHOLTZ FORCE ON A DIELECTRIC COATING FILM

The calculation of the Helmholtz force on the coating film may become ambiguous because of the delta-function contributions at the interfaces.

Since the Helmholtz force density is a surface force density and it is localized at the discontinuities of ε, the result for the total Helmholtz force will depend on whether the boundary between ε_1 and ε_2 at $z = d$ is included in the integration volume. One can define the force in two different ways, the first where the boundary is not included, and the second with the boundary included:

$$\langle \mathbf{F}_1^H\rangle = -\frac{1}{4} S \int_0^{d-\Delta} |E_x|^2 \frac{\partial \varepsilon}{\partial z}\, dz, \quad (66)$$

$$\langle \mathbf{F}_2^H\rangle = -\frac{1}{4} S \int_0^{d+\Delta} |E_x|^2 \frac{\partial \varepsilon}{\partial z}\, dz = \langle \mathbf{F}_1^H\rangle - \frac{1}{4} S \int_{d-\Delta}^{d+\Delta} |E_x|^2 \frac{\partial \varepsilon}{\partial z}\, dz, \quad (67)$$

where Δ is an infinitesimally small parameter. A finite contribution from the interface at $z = d$, given by the second term in Eq. (67), plays the role of a "surface tension," similar to the surface tension at the interface between liquid and gaseous phases. The surface contribution occurs as a result of the internal stress at such a boundary.

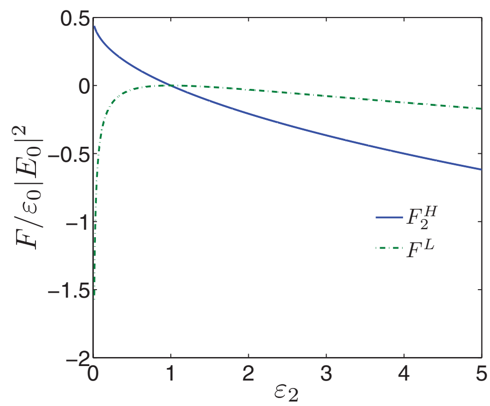


FIG. 5. (Color online) Force on a quarter-wavelength coating as a function of the dielectric constant from Eqs. (64) and (72).

After the integration, these forces become

$$\langle \mathbf{F}_1^H\rangle = -\frac{S\varepsilon_0|E_0|^2}{2}(\varepsilon_1 - 1) \times \left(\frac{\varepsilon_1 + \varepsilon_2 + (\varepsilon_1 - \varepsilon_2)\cos(2k_1 d)}{|g|^2}\right), \quad (68)$$

$$\langle \mathbf{F}_2^H\rangle = \langle \mathbf{F}_1^H\rangle - \frac{S\varepsilon_0|E_0|^2}{4}\frac{4(\varepsilon_2 - \varepsilon_1)\varepsilon_1}{|g|^2} \quad (69)$$

$$= \frac{S\varepsilon_0|E_0|^2}{2}(1 + |R|^2 - \varepsilon_2|T|^2). \quad (70)$$

Clearly these two expressions for Helmholtz force are different from each other and different from the Lorentz force in Eq. (63). For the case of the $\lambda/4$ coating, Eqs. (68) and (69) become

$$\langle \mathbf{F}_1^H\rangle = -\frac{S\varepsilon_0|E_0|^2}{4}(\sqrt{\varepsilon_2} - 1), \quad (71)$$

$$\langle \mathbf{F}_2^H\rangle = -\frac{S\varepsilon_0|E_0|^2}{2}(\sqrt{\varepsilon_2} - 1). \quad (72)$$

It is interesting to note that the Lorentz force expression for the $\lambda/4$ coating is always negative, while the Helmholtz force in Eq. (72) is negative for $\varepsilon_2 > 1$ and positive for $\varepsilon_2 < 1$. The effect of the boundary at $z = d$ is to double the amplitude of the Helmholtz force on the $\lambda/4$ layer. The forces on the $\lambda/4$ antireflective coating, calculated from Eqs. (64) and (72), are shown in Fig. 5.

The different expressions (64) and (72), resulting in different forces (and also in forces of a different sign), can potentially be tested experimentally. Another important difference is a finite and oscillating force density for the Lorentz force, Eq. (62), while the Helmholtz force has zero volume density. The oscillating force density of the Lorentz force would result in oscillating internal stresses, contrary to the uniform stress due to the Helmholtz force. This difference offers an interesting possibility of experimental verification. The net Helmholtz force is also different depending on whether the $\lambda/4$ layer is fused to the substrate or freely placed next to it (with an infinitely thin air gap). The experiment can also be conducted by measuring the force on the layer separated from the substrate by an air gap of finite width. We analyze this configuration in the next section.

VIII. THE FORCE ON A DIELECTRIC FILM SEPARATED FROM SUBSTRATE BY AN AIR GAP OF FINITE WIDTH

As it was noted in the previous section, due to surface contributions, the net Helmholtz force is different depending on whether the $\lambda/4$ layer is fused to the substrate or freely placed next to it. On the contrary, the net Lorentz force is the same independently on whether an infinitely thin air gap is introduced between the coating and the substrate. This can be viewed as a limit case a finite air gap. The latter configuration is also of interest for experimental studies.

Consider a slab of width d with dielectric constant ε_1, separated from a semi-infinite region of dielectric constant ε_2 by an air gap of width L. For this configuration, the electric field is given by

$$\begin{aligned} E_x &= (e^{ik_v z} + \mathrm{R}e^{-ik_v z})E_0 \quad z > 0, \\ E_x &= (\mathrm{A}e^{ik_1 z} + \mathrm{B}e^{-ik_1 z})E_0 \quad 0 < z < d, \\ E_x &= (\mathrm{C}e^{ik_v z} + \mathrm{D}e^{-ik_v z})E_0 \quad d < z < L + d, \\ E_x &= \mathrm{T}e^{ik_2 z} E_0 \quad L + d < z. \end{aligned} \tag{73}$$

It is not difficult to show that for this configuration, the force reduces to

$$\langle \mathbf{F}^L \rangle = \hat{\mathbf{z}}\tfrac{1}{2} S\varepsilon_0 |E_0|^2 \{1 + |R|^2 - (|C|^2 + |D|^2)\}. \tag{74}$$

This expression can be obtained by either direct integration of the Lorentz force density or by considering the total force on the body in air as given by (8). The general expression (74) is consistent with an intuitive corpuscular momentum balance similar to the one in Eq. (60). Indeed, the conservation of the momentum gives

$$\begin{aligned} F &= \hbar k\,[N_i - N_C + (N_R - N_D)] \\ &= \hbar k\, N_i\,(1 + |R|^2 - |C|^2 - |D|^2) \\ &= \tfrac{1}{2}\varepsilon_0 |E_0|^2 (1 + |R|^2 - |C|^2 - |D|^2), \end{aligned} \tag{75}$$

where the wave vector k is in vacuum, and $N_i = w/(\hbar\omega)$, $w = \varepsilon_0 |E_0|^2/2$.

The amplitude of the force oscillates according to the phase $k_1 d$ and $k_v L$. A plot of the total force on the slab as a function of the phase $k_v L$ due to the air gap is shown in Fig. 6. Depending on the value of $k_v L$ the force may become positive or negative, with periodicity of $\pi/2$, starting from $k_v L = \pi/4$. In the limit $L \to 0$, the force reduces to the negative value given by (64). In Fig. 7, the force on the dielectric layer is drawn as a function of the width of the layer for different values of the phase $k_v L$.

IX. THE FORCE ON A SEMI-INFINITE REGION AND ON A FINITE SLAB WITH DISSIPATION

It has already been noted in Sec. VI when analyzing the force on a coating, that the force density in the semi-infinite substrate was zero. This may lead to a number of superficial paradoxes.

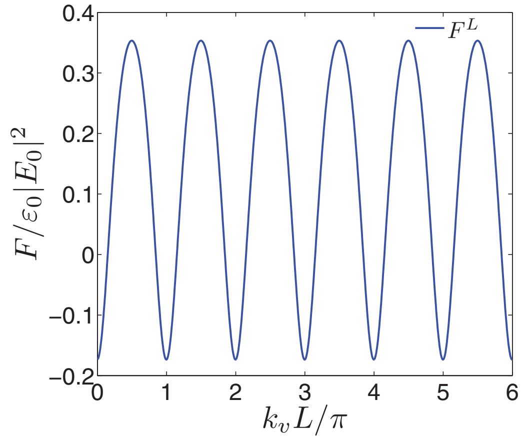


FIG. 6. (Color online) Total force on the dielectric layer as a function of the separation between the slab and the substrate; $\varepsilon_1 = 1.5$, $\varepsilon_2 = 2.25$ and $k_1 d = \pi/2$.

Let us calculate the force on a semi-infinite region by using the results of the previous section [e.g., the force on a semi-infinite region can be obtained by generalizing the result for the Lorentz force in Eq. (63)]. Setting $\varepsilon_1 = \varepsilon_2$ makes the coating and substrate equivalent so that whole system extends to infinity. In this case, $B = 0$, $A = T = 2/(1 + \sqrt{\varepsilon_2})$, $R = (1 - \sqrt{\varepsilon_2})/(1 + \sqrt{\varepsilon_2})$, and the Lorentz force from Eq. (63) becomes zero,

$$\langle \mathbf{F}^L \rangle = 0. \tag{76}$$

This result is consistent with a zero force density calculated using Eq. (53) (see also Fig. 4). The apparent paradox of this result is in the fact that for $\varepsilon_1 = \varepsilon_2 \neq 1$ there exists a wave reflected from the semi-infinite region and one expects a finite positive force applied to the semi-infinite dielectric. Another superficial paradox occurs with the Helmholtz force calculated for the semi-infinite region. Using Eq. (70) and setting $\varepsilon_2 = \varepsilon_1$ one finds the negative force [25,26],

$$\langle \mathbf{F}^H \rangle = -\hat{\mathbf{z}} \frac{\varepsilon_2 - 1}{(\sqrt{\varepsilon_2} + 1)^2} S\varepsilon_0 \,|E_0|^2, \tag{77}$$

applied to the dielectric boundary.

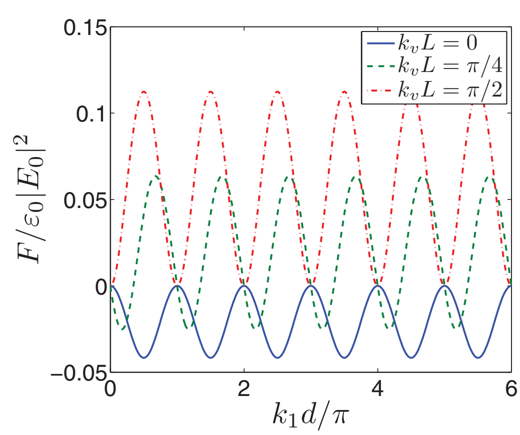


FIG. 7. (Color online) Total force on the dielectric layer as a function of the width of the layer for different values of the distance between the slab and the substrate.

At first sight, the results in Eq. (76) and in Eq. (77) are obviously inconsistent with each other. This inconsistency occurs due to the nonregular nature of the semi-infinite dielectric case without dissipation due to failure to properly account for the wave momentum at infinity. This is also apparent from the expression in Eq. (57): This expression remains oscillatory for any $d \to \infty$. A finite length slab without dissipation cannot be extended to infinity without explicit specification of the boundary condition for the wave amplitude (and the wave momentum) at the opposite end of the slab.

The natural way used to deal with the $z \to \infty$ behavior in a semi-infinite region is to assume an infinitely small dissipation in the dielectric [7,8,11,23,24]. In a semi-infinite medium, even with an infinitesimally small dissipation, all the momentum transmitted to the dielectric will be absorbed, thus generating a finite force on the dielectric, while the force density goes to zero. Such calculations can easily be done starting from the case of a finite length slab with dissipation, with vacuum regions on both sides. Assuming that the wave vector and the dielectric constant are complex such that $k = k_r + i k_i$ and $\varepsilon = \varepsilon_r + i\, \varepsilon_i$, the Lorentz force density is given by

$$\langle \mathbf{f}^L \rangle = \hat{\mathbf{z}} \frac{\varepsilon_0}{2} \{ (k_i - \varepsilon_r k_i + \varepsilon_i k_r)(|A|^2 e^{-2k_i z} - |B|^2 e^{2k_i z}) - 2(k_r - \varepsilon_i k_i - \varepsilon_r k_r) \mathrm{Im}(A^* \mathrm{B} e^{-2ik_r z}) \}. \tag{78}$$

The force density is localized in the region of the order of k_i^{-1} and decays away from the boundary. Note that in the absence of dissipation ($k_i = \varepsilon_i = 0$), Eq. (78) becomes equivalent to Eq. (54). The Lorentz force on the slab with dissipation is then obtained by integrating Eq. (78) over the length of the slab. The result of the integration for a slab of finite length with dissipation can be written in the form,

$$\langle \mathbf{F}^L \rangle = \hat{\mathbf{z}} \frac{S \varepsilon_0 |E_0|^2}{2} (1 + |R|^2 - |T|^2). \tag{79}$$

This expression is similar to the one obtained for the case of the nondissipative slab, Eq. (57), however, in the presence of dissipation, the equality $|R|^2 + |T|^2 = 1$ is no longer valid, where T is the transmission coefficient defined for the wave transmitted into the vacuum region on the right of the slab.

It is important to note that the expression in Eq. (17) for the Helmholtz force is valid only for the case of real ε and μ and thus cannot be used to analyze the dissipative case. We need to generalize the expression for the Helmholtz force to account for finite dissipation. From (16) one finds

$$f_i^H = \frac{1}{2} D_j \frac{\partial}{\partial x_i} E_j - \frac{1}{2} E_j \frac{\partial}{\partial x_i} D_j. \tag{80}$$

Assuming, as for the Lorentz force, a complex dielectric constant in the form $\varepsilon = \varepsilon_r + i\, \varepsilon_i$ and the constitutive relation $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$, the time average of Eq. (80) reduces to

$$\langle f_z^H \rangle = -\frac{1}{2} \varepsilon_i \, \mathrm{Im} \left(E_x \frac{\partial}{\partial z} E_x^* \right) - \frac{1}{4} |E_x|^2 \frac{\partial \varepsilon_r}{\partial z}. \tag{81}$$

It follows from (81) that in the general case of a complex ε, in addition to the surface part that depends on the real part of the dielectric constant, the Helmholtz force density acquires a bulk (volume) part that depends on the imaginary part of the dielectric constant.

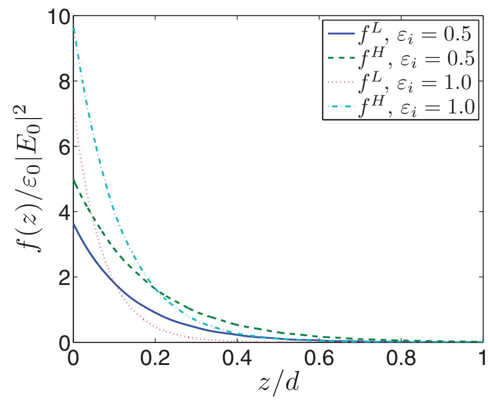


FIG. 8. (Color online) Lorentz force density and volume part of the Helmholtz force density for the case of a dissipative slab as given by Eqs. (78) and (81); $\varepsilon_r = 2.25$.

The cumulative integral of the Helmholtz force density in the dissipative case, given by

$$\langle \mathbf{F}^H(z) \rangle = \hat{\mathbf{z}} S \int_0^z f^H(z') dz', \tag{82}$$

has jumps at the points $z = 0$ and $z = d$, which correspond to the effects of the surface contributions in the Helmholtz force density as can be seen in Fig. 9.

The total force on the dissipative slab is calculated by integrating the expression in Eq. (81) over the length of the slab. By solving for the fields inside the slab and applying boundary conditions, the Helmholtz force on a slab with dissipation immersed in vacuum can be shown to be equal to

$$\langle \mathbf{F}^H \rangle = \hat{\mathbf{z}} \frac{S \varepsilon_0 |E_0|^2}{2} (1 + |R|^2 - |T|^2). \tag{83}$$

An important result is that this expression is identical to the total Lorentz force, Eq. (79). It is worth noting that while the total force applied to the slab are the same in both formulations, the spatial profiles for the Lorentz force and Helmholtz expressions are dramatically different. A comparison of the Lorentz force density and the volume part of the Helmholtz force density for the dissipative slab is shown in Fig. 8. Note

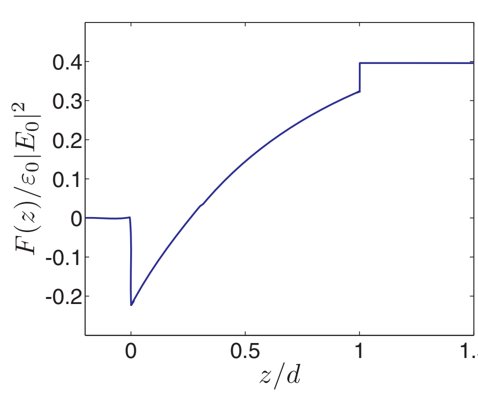


FIG. 9. (Color online) Integral of Helmholtz force density in a dissipative slab as given by Eq. (82); $\varepsilon = 2.25 + 0.1i$.

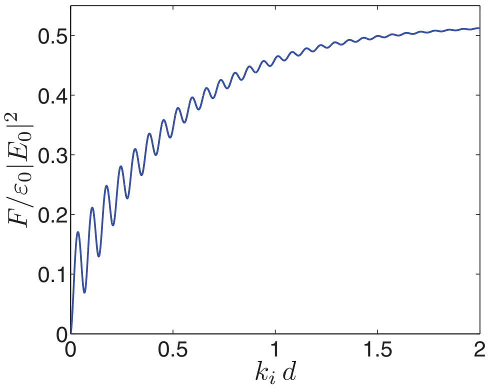


FIG. 10. (Color online) Force per unit area on a dissipative slab as given by Eqs. (79) and (83); $\varepsilon = 2.25 + 0.1i$. As can be seen, for $k_i d \gg 1$, the force tends to a constant value.

that the Helmholtz force has additional surface contribution not shown Fig. 8.

An example of the total force on a slab with dissipation as calculated in Eqs. (79) and (83) is given in Fig. 10 as a function of the slab length. At small $k_i d$, the force amplitude oscillates and saturates to a constant value for large $k_i d$.

In the limit of large length (or strong dissipation), when $k_i d \gg 1$, we have that $|T|^2 \to 0$ and the problem of a finite slab with dissipation and a semi-infinite medium (with infinitely small dissipation) are equivalent. In this case, Eqs. (79) and (83) give [7]

$$\langle \mathbf{F}^L \rangle = \langle \mathbf{F}^H \rangle = \hat{\mathbf{z}} \frac{S\varepsilon_0 |E_0|^2}{2}(1 + |R|^2). \tag{84}$$

This force corresponds to the momentum of the electromagnetic field transmitted into the dielectric and eventually absorbed over the large (infinite) length. When the dissipation is absent, the wave momentum is carried to infinity. The expression in Eq. (84) is consistent with simple corpuscular momentum balance similar to (60), assuming that all the momentum inside the medium is eventually absorbed at infinity. The conservation of the momentum gives

$$\begin{aligned} F &= \hbar k\,[N_i + N_R] \\ &= \hbar k\, N_i\,(1 + |R|^2) \\ &= \tfrac{1}{2}\varepsilon_0 |E_0|^2 (1 + |R|^2), \end{aligned} \tag{85}$$

where the wave vector k is in vacuum, and $N_i = w/(\hbar\omega)$, $w = \varepsilon_0 |E_0|^2/2$.

X. DISCUSSION AND SUMMARY

Maxwell equations lead to several possible forms of the conservation equation for a momentumlike quantity, as in Eqs. (14), (18), (28), and (41). All these forms are mathematically equivalent within the Maxwell macroscopic electrodynamics. However, the separation between the material and field parts of the momentum is not always clearly defined. As a result, the exchange term in these equations is not clearly identifiable as the density of the electromagnetic force acting on the medium. It is shown here that different expressions for the force density with different structure have substantially different spatial profiles (e.g., the Helmholtz force density given by Eq. (17)], in the absence of dissipation, is exclusively a surface force density and is localized at the boundaries, while the Lorentz force density given by Eq. (31) is a oscillating volume force density as can be seen in Fig. 1. It has been shown here that different expressions for the force density correspond to different assumptions on the averaging length scale inherently assumed in any macroscopic model.

The identical transformations between different forms of the momentum conservation (respectively, between different forms of the force density) involve transformations between volume and surface terms. The volume terms in the force density correspond to the internal stress (pressure) forces. Volume averaging leads to cancellation of internal forces so that only the surface (with respect to the averaging volume size) remains. Obviously, the assumed ordering of the averaging length scales and the length scales for the electric charge and electromagnetic field distribution affect the structure of the internal forces and eventually the expressions for the force density. Such difference is demonstrated explicitly via the direct averaging of microscopic forces as in Sec. IV. This derivation is also valid for a dispersive medium, as in plasma.

Two different expressions for the force density, dipole force (40) and Lorentz force (31), are derived directly in Sec. IV by making different assumptions on involved length scales for the electromagnetic field and charge distribution. For the transverse electromagnetic waves, these two expressions are identical and produce the force density that follows a standing wave pattern inside the body as it is illustrated in Fig. 1. It is worth noting here that, on the contrary, the Helmholtz force density for a nondissipative media is given by delta-function surface force. One can argue therefore that the Helmholtz force density corresponds to the averaging sample of the size of the whole body. While the total force acting on a body immersed in vacuum is the same for different forms of the momentum conservation, the difference in force density profiles and in internal stresses as discussed in this paper, in principle, can be measured experimentally. The experimental measurements of the internal stress (together with the mechanical stress properties of the material) may provide the information on the actual force density. The measurements conducted at different resolutions can be used to confirm the point of view advocated in this paper: Different expressions for the force density exist at different length scales corresponding to the different averaging (sampling) volumes.

The total force on the dielectric plane slab was calculated in this paper for several configurations. For a single dissipative slab, this force is given by

$$F = \frac{\varepsilon_0 |E_0|^2}{2}(1 + |R|^2 - |T|^2). \tag{86}$$

Here T is the wave transmission coefficient for the outgoing wave in vacuum region on the right outside of the slab. This expression applies both for dissipative and nondissipative cases. We have shown that the same result can be obtained from the Helmholtz force generalized to the dissipative case. In the nondissipative case, the condition $|R|^2 + |T|^2 = 1$ can be used, so that the force reduces to the expression $F = \varepsilon_0 |E_0||R|^2$. Equation (86) is valid for arbitrary properties of the plane slab dielectric, including those of metamaterials and negative refraction media. The net force always remains positive (pushes the slab away).

Equation (86) can also be used to determine the force on a semi-infinite region. In a semi-infinite case, one needs to specify the fate of the momentum flux at infinity. For any realistic conditions, even an infinitesimal dissipation will lead to an eventual damping of the outgoing wave in the semi-infinite region, so that $T = 0$ in Eq. (86). The resulting net force $F = \varepsilon_0|E_0|(1 + |R|^2)/2$ is consistent with the corpuscular momentum balance as discussed in Sec. IX: The total force is due to the absorption of the net wave momentum in the medium. It is worth noting that the force density is infinitesimally small for vanishing (but finite) dissipation, while the integral for the total force remains finite. We have generalized the expression for the Helmholtz force density to include dissipation and have shown that the resulting Helmholtz force is identical to the Lorentz force. In addition to the negative delta-function surface contribution at the boundary, the Helmholtz force density has also a bulk contribution, which is positive, so the net force becomes positive as shown in Fig. 10. Without dissipation, the Helmholtz force has only a negative part localized at the boundary, arising from the break in the homogeneity of the space caused by the interface (the discontinuity in ε). The fact that this inhomogeneity is accompanied by a surface force is a clear indication that the Helmholtz force corresponds to the conservation of a pseudomomentum [27]. The striking difference in the force density profiles of the Lorentz and Helmholtz forces would result in different distributions of the internal stresses that can be potentially detected in experiment.

We have calculated the total force on the dielectric slab coating as well as in the case of the slab separated from the substrate by a finite air gap. For this case, the force amplitude oscillates and can be both positive or negative depending on the slab width and distance from the substrate. This effect may possibly be used for aggregation and separation of nanoparticles.

ACKNOWLEDGMENTS

This work was supported by Natural Sciences and Engineering Research Council of Canada. Stimulating discussions with A. Hirose are acknowledged with gratitude.

[1] A. Ashkin, Phys. Rev. Lett. **24**, 156 (1970).
[2] A. Ashkin, J. M. Dziedzic, J. E. Bjorkholm, and Steven Chu, Opt. Lett. **11**, 288 (1986).
[3] A. Ashkin and J. M. Dziedzic, Science **235**, 1517 (1987).
[4] G. Volpe, R. Quidant, G. Badenes, and D. Petrov, Phys. Rev. Lett. **96**, 238101 (2006).
[5] Y. Harada and T. Asakura, Opt. Commun. **124**, 529 (1996).
[6] K. Ladavac and D. Grier, Opt. Express **12**, 1144 (2004).
[7] M. Mansuripur, Opt. Express **12**, 5375 (2004).
[8] S. M. Barnett and R. Loudon, Philos. Trans. R. Soc. London A **368**, 927 (2010).
[9] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Rev. Mod. Phys. **79**, 1197 (2007).
[10] J. P. Gordon, Phys. Rev. A **8**, 14 (1973).
[11] B. Kemp, T. Grzegorczyk, and J. Kong, Opt. Express **13**, 9280 (2005).
[12] I. Brevik, Phys. Rep. **52**, 133 (1979).
[13] W. She, J. Yu, and R. Feng, Phys. Rev. Lett. **101**, 243601 (2008).
[14] Zhong-Yue Wang, Pin-Yu Wang, and Yan-Rong Xu, Optik **122**, 1994 (2011).
[15] H. Minkowski, Nach. Ges. Wiss. Gotingen **1**, 53 (1908).
[16] M. Abraham, Rendiconti del Circolo Matematico di Palermo (1884–1940) **28**, 1 (1909).
[17] G. Russakoff, Am. J. Phys. **38**, 1188 (1970).
[18] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[19] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, New York, 1962).
[20] A. D. Yaghjian, IEEE Trans. Antennas Propag. **55**, 1495 (2007).
[21] J. H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities* (Oxford University Press, Oxford, 1965).
[22] K. Oughstun, *Electromagnetic and Optical Pulse Propagation 1: Spectral Representations in Temporally Dispersive Media* (Springer, New York, 2006).
[23] R. Loudon, S. M. Barnett, and C. Baxter, Phys. Rev. A **71**, 063802 (2005).
[24] S. M. Barnett and R. Loudon, J. Phys. B **39**, S671 (2006).
[25] A. Hirose and R. Dick, Can. J. Phys. **87**, 407 (2009).
[26] A. Hirose, Can. J. Phys. **88**, 247 (2010).
[27] R. Peierls, Sov. Phys. Usp. **34**, 817 (1991).