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NONLINEAR PHENOMENA

Nonlinear Damping of Zonal Flows[1]

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Abstract—The modulatonal instability theory for the generation of large-scale (zonal) modes by drift modes has been extended to the second order including the effects of finite amplitude zonal flows, ϕ_q. The nonlinear (second-order) sidebands are included in the perturbative expansion to derive the nonlinear equation for the evolution of ϕ_q. It is shown that effects of finite ϕ_q reduce the growth rate of zonal flow with a possibility of oscillatory regimes at a later stage.

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1. INTRODUCTION

Drift waves (DWs) and instabilities are common for many confined plasmas. Nonlinear interactions of drift waves have been studied in various settings in attempts to understand anomalous transport in controlled fusion systems such as tokamaks. The nonlinear Hasegawa–Mima equation is often used as a simplest model for drift waves and generation of large-scale structures such as zonal flows (ZFs). Similar phenomena occurs in geostrophic fluids (shallow water on a rotating sphere) such as the atmosphere and ocean, where the analogous Charney–Obukhov equation is employed to describe Rossby waves. Zonal flow structures have been a topic of intense interest due to their role in controlling the drift turbulence by taking energy away from small-scale fluctuations as well by a direct mechanism via the reduction of the radial correlation length [2].

The basic dynamics in drift wave–zonal flow systems can be characterized by a predator–pray model [12, 18], where drift waves are the pray, while zonal flow is the predator who "feeds" on drift waves. In this model, the evolution of zonal flow energy is described by the equation

$$\frac{\partial W_{\rm ZF}}{\partial t} = \kappa W_{\rm DW} W_{\rm ZF} - \gamma_{\rm damp} W_{\rm ZF} - \gamma_{\rm NL}(W_{\rm ZF}) W_{\rm ZF}, \quad (1)$$

where $W_{\rm ZF}$ and $W_{\rm DW}$ are zonal flow and drift wave energy, respectively.

The first term on the right side of Eq. (1) describes nonlinear coupling between drift wave and zonal flow. This coupling is manifested as a zonal flow instability which has a growth rate proportional to the drift wave intensity, $\gamma = \kappa W_{\rm DW}$. The modulational instability theory of drift waves is the simplest model that describes zonal flow growth [1, 3, 4, 8, 13–16]. Such analytical calculations are generally consistent with the results of direct numerical simulations [8, 10, 11]. The second term in Eq. (1) describes the linear ZF damping rate, e.g., collisional or neoclassical nature [18]. The last term in Eq. (1) describes nonlinear dumping of ZF. In the simplest case, it can be represented in the form $\gamma_{\rm NL} = \alpha W_{\rm ZF}$, where α is the so-called Landau constant [6]. Nonlinear damping may suppress the zonal flow instability. Eq. (1) gives a simple estimate for the zonal flow energy (amplitude) at saturation,

$$W_{\rm ZF}^{\rm max} \sim \frac{\gamma}{\alpha}. \quad (2)$$

Several different mechanisms resulting in nonlinear damping of zonal flow are possible (e.g., effect of broad drift wave spectra, secondary instabilities of zonal flows, interaction with mean flow (MF) [17], etc. [18]). In this work we focus on generalizing modulation instability theory for case of finite ZF amplitude by nonlinear modification of Reynolds stress tensor drive. Generally speaking, this effect is a nonlinear expansion of the coupling with drift waves (first term in Eq. (1)),

$$\kappa = \kappa_0 + \kappa_1 W_{\rm ZF} + O(W_{\rm ZF}^2). \quad (3)$$

Similar studies were conducted by Mendonca [9] using the wave kinetic equation. Here we employ the direct perturbation theory for several coupled modes.

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The physical nature of considered nonlinear damping is ZF interaction with itself. In some sense (mathematically) our ZF–ZF interaction is similar to stabilization via ZF–MF interaction considered by K. Uzawa et al. [17]. Uzawa concluded that taking into account leading stabilization term is sufficient, and one does not need to include higher side-bands. In our system the leading stabilizing effect is due to self ZF interaction. Moreover, in real systems, the importance of ZF–MF interaction against ZF–ZF interaction would be determent by ZF/MF energy balance.

The paper is organized as follows. In Section 2, we introduce simple modulation instability theory of ZF/DW to derive the linear growth rate ($\gamma = \kappa W_{\rm DW}$) in Eq. (1). In Section 3, we extend the results of Section 2 for the case of finite ZF amplitude by taking into account the second-order sidebands. This allows us to estimate saturation amplitude Eq. (2) and nonlinear damping coefficient ($\alpha = \gamma_{\rm NL}/W_{\rm ZF}$). We provide a summary of the manuscript results in Section 4.

2. DRIFT WAVES–ZONAL FLOWS INTERACTIONS IN HASEGAWA–MIMA MODEL

To derive coupling coefficient (κ_0 in Eq. (3)) from modulational instability theory, we use Fourier decomposition of the standard Hasegawa–Mima equation [5],

$$D_k(\omega)\varphi_k(\omega) + \sum_{k=k'+k''} B_{k',k'}\varphi_{k'}(\omega')\varphi_{k''}(\omega'') = 0, \quad \text{(4a)}$$

$$D_k(\omega) = -i\omega(1+\rho_s^2k^2) + i\mathbf{V}_*\cdot\mathbf{k}, \quad \text{(4b)}$$

$$B_{k',k''} = \frac{c\rho_s^2}{B_0}(\hat{\mathbf{z}}\cdot\mathbf{k}'\times\mathbf{k}'')(k'^2 - k''^2), \quad \text{(4c)}$$

where ϕ_k is the Fourier transform of electrostatic potential corresponding to the $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ mode (here and later, the ω dependence is omitted for convenience), $\mathbf{V}_* = V_*\hat{\mathbf{y}}$ is the electron diamagnetic drift velocity, ρ_s is the gyroradius, c is the speed of light, and B_0 is the stationary magnetic field.

Nonlinear part of Eq. (4) is a sum of three-wave interactions. The linear stage of zonal flow instability is obtained by truncating nonlinear part of Eq. (4) and including only the primary drift wave $(\omega, \mathbf{k})$ mode, the zonal flow $(\Omega, \mathbf{q})$ mode, and two sidebands $(\Omega\pm\omega, \mathbf{q}\pm\mathbf{k})$ modes. The fact that electrostatic potential is observable physical quantity implies this constrain which we will use later,

$$\phi_j^* = \phi_{-j}, \mathbf{k}_{-j} = -\mathbf{k}_j.$$

Then, the electrostatic potential is represented in the form

$$\phi(t,\mathbf{r}) = (\Phi_k^\omega + \text{c.c.}) + \Phi_q^\Omega + \Phi_{q+k}^{\Omega+\omega} + \Phi_{q-k}^{\Omega-\omega}, \quad (5)$$

where

$$\Phi_k^\omega = \phi_k e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}.$$

Basically truncation (5) is a first-order perturbation expansion with ZF amplitude as a small parameter ($\phi_q \ll \phi_k$). This is true because side-bands amplitude is proportional to the ZF amplitude, or more generally $|\phi_{nq+k}| \sim |\phi_q|^n$. To obtain the dispersion equation for the ZF in this limit, we substitute the truncated form of electrostatic potential (5) in Eq. (4). Thus, evolution equations for ϕ_q and $\phi_{q\pm k}$ are

$$D_q\phi_q + B_{k,q-k}\phi_k\phi_{q-k} + B_{-k,q+k}\phi_{-k}\phi_{q+k} = 0, \quad (6)$$

$$D_{q\pm k}\phi_{q\pm k} + B_{\pm k,q}\phi_{\pm k}\phi_q = 0, \quad (7)$$

where

$$D_q = -i\Omega(1+\rho_s^2q^2),$$
$$D_{q\pm k} = -i\big[(\Omega\pm\omega) - \mathbf{V}_*\cdot(\mathbf{q}\pm\mathbf{k}) + \rho_s^2(\Omega\pm\omega)(\mathbf{q}\pm\mathbf{k})^2\big],$$
$$B_{\pm k,q} = \pm\frac{c\rho_s^2}{B_0}\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{q}(k^2-q^2).$$

In the leading order, the primary wave amplitude does not change, giving the linear dispersion equation for drift wave ($D_k = 0$),

$$\omega = \frac{V_*k_y}{1+\rho_s^2k^2}. \quad (8)$$

Eliminating sidebands amplitudes ($\phi_{q\pm k}$) from Eqs. (6) and (7), we have

$$D_q = |\phi_k|^2\left[\frac{B_{k,q-k}B_{-k,q}}{D_{q-k}} + \frac{B_{-k,q+k}B_{k,q}}{D_{q+k}}\right], \quad (9)$$

where

$$B_{\pm k,q\mp k} = \frac{c\rho_s^2}{B_0}\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{q}(2\mathbf{q}\cdot\mathbf{k}\mp q^2).$$

The explicit form of the dispersion equation can be written as

$$\left[\Omega(1+Q) - qV_{gx}\right]^2 = \left[\omega Q - \frac{\Omega}{\omega}qV_{gx}\right]^2 - 2\omega_{ci}^2\left|\frac{e\phi_k}{T_e}\right|^2 k_y^2q^4(k^2-q^2)\rho_s^8K, \quad (10)$$

where

$$V_{gx} = -\frac{2\omega\rho_s^2k_x}{1+\rho_s^2k^2}, \quad Q = \frac{q^2\rho_s^2}{1+k^2\rho_s^2},$$
$$K = \frac{1+\rho_s^2k^2+\rho_s^2q^2-4\rho_s^2k_x^2}{(1+\rho_s^2q^2)(1+\rho_s^2k^2)^2},$$

and $\omega_{ci} = eB_0/m_i c$ is the ion gyrofrequency, m_i is the ion mass, e is the electron charge, and T_e is the electron temperature.

In the long-wavelength limit ($k\rho_s \ll 1$ and $q\rho_s \ll 1$) and when $\mathbf{q}\cdot\mathbf{k} = 0$, the solution to Eq. (10) have a simple form

$$\Omega^2 = q^4\rho_s^4\left(\omega^2 - 2\left|\frac{e\varphi_k}{T_e}\right|^2 k^4 c_s^2\rho_s^2\right). \quad (11)$$

This equation shows that the zonal flow instability occurs for a sufficiently large amplitude of the primary drift wave (see Malkov [7] and others [8, 13]),

$$\left|\frac{e\phi_k}{T_e}\right| > \frac{1}{kL_n}, \quad (12)$$

where L_n is a scale of density change ($L_n = n_0/|\nabla n_0| \sim c_s\rho_s/V_*$) and $c_s = T_e/m_i$ is the ion sound velocity. Note that the amplitude threshold in Eq. (11) is somewhat equivalent to the linear damping term $\gamma_{\rm damp}$ in Eq. (1). It is interesting that the threshold amplitude of the unstable primary wave is of the order of the mixing length amplitude.

3. EFFECTS OF FINITE AMPLITUDE OF ZONAL FLOW

The leading order of perturbation expansion of ZF frequency (growth rate) Eq. (11) does not depend on the ZF amplitude. In this section, we derive this dependence with second-order perturbation expansion. The second-order term is a nonlinear self damping of ZF (the third term in Eq. (1)). To do so, we are extending model from Section 2 by including second-order sidebands ($\pm 2\mathbf{q} \pm \mathbf{k}$),

$$\begin{aligned}\phi(t,\mathbf{r}) &= (\Phi_k^{\omega} + \text{c.c.}) + (\Phi_q^{\Omega} + \text{c.c.}) \\ &+ (\Phi_{q+k}^{\Omega+\omega} + \text{c.c.}) + (\Phi_{q-k}^{\Omega-\omega} + \text{c.c.}) \\ &+ \Phi_{2q+k}^{2\Omega+\omega} + \Phi_{2q-k}^{2\Omega-\omega}.\end{aligned} \quad (13)$$

It is worth noting, that the main assumption ($\phi_q \ll \phi_k$) is still holds as we omitted higher order sidebands (e.g., $\pm 3\mathbf{q} \pm \mathbf{k}$). Repeating the procedure form Section 2, the dispersion equation is obtained as

$$\begin{aligned}D_q &= |\phi_k|^2\left[\frac{B_{k,q-k}B_{-k,q}}{D_{q-k}} + \frac{B_{-k,q+k}B_{k,q}}{D_{q+k}}\right] \\ &+ |\phi_q|^2|\phi_k|^2\left[\frac{B_{-q+k,2q-k}B_{k,-q}B_{q-k,q}B_{q,-k}}{D_{-q+k}D_{2q-k}D_{q-k}}\right. \\ &\left.+ \frac{B_{-q-k,2q+k}B_{-k,-q}B_{q+k,q}B_{q,k}}{D_{-q-k}D_{2q+k}D_{q+k}}\right],\end{aligned} \quad (14)$$

where

$$D_{2q+k} = D_{q+k}(\Omega \to 2\Omega, \mathbf{q} \to 2\mathbf{q}),$$

$$B_{\pm k,-q} = -B_{\pm k,q} = \mp\frac{c\rho_s^2}{B_0}\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{q}(k^2 - q^2),$$

$$B_{q,q\pm k} = \frac{c\rho_s^2}{B_0}\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{q}(2\mathbf{q}\cdot\mathbf{k} \pm k^2),$$

$$B_{-q\pm k,2q\mp k} = \frac{c\rho_s^2}{B_0}\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{q}(2\mathbf{q}\cdot\mathbf{k} \mp 3q^2).$$

Equation (14) differs from simplified equation (9) by the additional part which is quadratic in respect to ZF amplitude ($|\phi_q|$). The last bracket of Eq. (14) resembles "α" Landau constant, which implies that the sign of the term in the last bracket governs saturation, while the ratio of the terms in the first and second brackets defines the saturation amplitude.

The explicit dispersion equation is cumbersome in this limit, so we leave only main terms in $\Omega, \mathbf{q}$-Taylor series. This is justified because zonal flow does not exist in linear limit ($\Omega_{\rm lin} = 0$) and are induced only via nonlinear interactions with drift waves, so $\Omega \ll \omega$ and $q \ll k$:

$$\begin{aligned}\Omega &= \frac{2\omega_{ci}^2\Omega\left|\frac{e\phi_k}{T_e}\right|^2 q^4k_y^2(k^2 - q^2)\rho_s^8 K}{\left(\omega Q - \frac{\Omega}{\omega}qV_{gx}\right)^2 - \left(\Omega(1+Q) - qV_{gx}\right)^2} \\ &+ \frac{\omega_{ci}^4\left|\frac{e\phi_k}{T_e}\right|^2\left|\frac{e\phi_q}{T_e}\right|^2 q^6k_y^4k^4\rho_s^{14}\left[\Omega M - qV_{gx}L\right]}{(\Omega - qV_{gx})^4},\end{aligned} \quad (15)$$

where

$$M = \frac{\left(16k_x^2k^2\rho_s^4 - \rho_s^2(1+\rho_s^2k^2)(3k^2 + 4k_x^2)\right)}{(1+\rho_s^2q^2)(1+\rho_s^2k^2)^4},$$

$$L = \frac{\rho_s^2(k^2 - 4k_x^2)}{(1+\rho_s^2q^2)(1+\rho_s^2k^2)^3}.$$

As before, we consider the long wavelength limit and the case when ZF propagate perpendicular to the primary wave ($\mathbf{q}\cdot\mathbf{k} = 0$). In this situation, the solution of Eq. (15) simplifies significantly. Considering the primary wave above the threshold (Eq. (11)), one writes

$$\begin{aligned}(\Omega_\pm/\omega_{ci})^2 &= -\left|\frac{e\phi}{T_e}\right|^2 k^4q^4\rho_s^8 \\ &\pm\sqrt{\left|\frac{e\phi_k}{T_e}\right|^4 k^8q^8\rho_s^{16} - 3\left|\frac{e\phi_q}{T_e}\right|^2\left|\frac{e\phi_k}{T_e}\right|^2 k^{10}q^6\rho_s^{16}}.\end{aligned} \quad (16)$$

The solution with negative sign (corresponding to the ZF instability) can be expanded giving

$$(\Omega_-/\omega_{ci})^2 \simeq -2\left|\frac{e\phi_k}{T_e}\right|^2 k^4q^4\rho_s^8 + \frac{3}{2}\left|\frac{e\phi_q}{T_e}\right|^2 k^6q^2\rho_s^8. \quad (17)$$

This equation shows that finite amplitude ϕ_q results in stabilization of ZF instability. The amplitude of stabilized ZF is of the order

$$\left|\frac{e\phi_q}{T_e}\right|_{\max} \sim \frac{q}{k}\left|\frac{e\phi_k}{T_e}\right|. \tag{18}$$

Strictly speaking, this value is at the limit of applicability of the perturbation expansion ($\phi_q \ll \phi_k$). However, taking into account that $q \ll k$, Eq. (18) yields the main assumption $\phi_q \ll \phi_k$. Thus, one can expect that the above estimate is still valid as an order of magnitude estimate. In Eq. (18) regime, ZF dynamics becomes oscillatory (with $\Re(\Omega) \neq 0$) which is common in numerical ZF simulations.

It was pointed out Manfredi [8] that, at some point, the amplitude of DW is starting to decrease resulting in saturation of ZF growth. It is possible to estimate ZF amplitude using this considerations (see J. Anderson et al. [13]). However, this effect will be important when the amplitude of ZF is comparable to DW amplitude ($|\phi_q| \sim |\phi_k|$) and is not considered in our paper. Simulation results that J. Anderson et al. [13] used to support their estimate that $\phi_q\gamma_q$ grows as ϕ_k^2 agree with our results, because γ_q growth as ϕ_k, so ϕ_q grows as ϕ_k as in Eq. (18).

4. SUMMARY

In this article, we discussed the evolution of zonal flow in framework of drift wave turbulence model described by Hasegawa–Mima equation. Within the qualitative picture of the predator-prey model, Eq. (1), zonal flow dynamics is governed by the competition of the zonal instability and nonlinear saturation. The focus of our work was on the derivation of nonlinear damping term via direct perturbation theory.

The dispersion of drift waves results in the amplitude threshold in Eq. (11) effectively equivalent to the linear damping term in zonal flow model equation (1). The nonlinear damping of zonal flow (the last term in Eq. (1)) was obtained by expanding the coupling to higher order, Eq. (3), by including effects of finite amplitude of ZF. Nonlinear dispersion equation for zonal flow instability (15) was derived. It is shown that, in the long wavelength limit, the nonlinear effects stabilize zonal flow growth. The estimate for the maximum ZF amplitude was obtained (Eq. (18)).

It is understood that ZF is important in the Dimits shift formation process. Thus, one can envisage that the stabilization mechanism due to a finite amplitude of ZF flow may be operative and shift the instability boundary. However, it is really speculative, since we do not consider the really unstable modes (such as ITG) mode. Our model is based on the Hasegawa–Mima equation, for the conditions of the tokamak, the zonal flow with $m = 0$, will not follow the Boltzmann distribution for ions, so the Hasegawa–Mima equation should be modified [14]. In that analysis has to be modified [14] and the $k^2\rho^2$ will be different (smaller) see [8, 14].

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