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Nonlinear excitation of long-wavelength modes in Hall plasmas

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Hall plasmas with magnetized electrons and unmagnetized ions exhibit a wide range of small scale fluctuations in the lower-hybrid frequency range as well as low-frequency large scale modes. Modulational instability of lower-hybrid frequency modes is investigated in this work for typical conditions in Hall plasma devices such as magnetrons and Hall thrusters. In these conditions, the dispersion of the waves in the lower-hybrid frequency range propagating perpendicular to the external magnetic field is due to the gradients of the magnetic field and the plasma density. It is shown that such lower-hybrid modes are unstable with respect to the secondary instability of the large scale quasimode perturbations. It is suggested that the large scale slow coherent modes observed in a number of Hall plasma devices may be explained as a result of such secondary instabilities. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4964724]

I. INTRODUCTION

Oscillations in a wide range of frequencies from 10 kHz to 500 MHz observed in Hall thrusters (see, e.g., Refs. 1 and 2) are inherent to these devices and can play a decisive role in setting the performance and efficiency by regulating the acceleration processes and the anomalous transport (in particular, the anomalous electron mobility).[1,3] Magnetron devices, operating in similar physical parameter regimes,[4] also exhibit a range of fluctuations and self-organized structures.[5]

Theoretical studies revealed the existence of a number of linear plasma instabilities which are considered as the sources of fluctuations and turbulence in Hall plasma devices. One of the main features of these devices is the presence of stationary, externally applied electric field $\mathbf{E}_0$, which is perpendicular to the external equilibrium magnetic field $\mathbf{B}_0$. The strength of the magnetic field is chosen such that electrons are magnetized, $\rho_e \ll L$, and ions are not, $\rho_i \gg L$, where L is the characteristic length scale of the plasma region and $\rho_{e,i}$ are electron and ion Larmor radii. Such conditions are defined here as Hall plasma regime. This configuration results in the equilibrium $\mathbf{E}_0 \times \mathbf{B}_0$ electron drift which is a source of free energy for a number of instabilities.[6,7] In combination with gradients of the magnetic field and plasma density, the electron drift results in the instability which was identified experimentally and studied theoretically.[8,9] Recently, a linear theory of this instability was revisited in Ref. 10 correctly taking into account, compressibility of the electron flow in inhomogeneous magnetic field and the electron temperature perturbations. Later theoretical studies revealed the other mechanisms due to electron collisions with neutral atoms and plasma ionization.[11–13] The reviews of different kinds of oscillations and instabilities in Hall thrusters with the corresponding references to original works are given in several references such as Refs. 1, 2, and 13.

Among the various fluctuations observed in Hall thrusters, the low-frequency large scale structures, called spokes or spikes, have long attracted the interest since initial experimental studies in the 1960s and the 1970s. Despite a number of publications devoted to theoretical studies of oscillations and instabilities, the nature of low-frequency long wavelength structures is not well understood. Here we suggest a new mechanism of instability which can be responsible for long-wavelength, low-frequency oscillations observed in Hall plasma devices. The proposed mechanism is based on the modulational instability of high-frequency, short-wavelength lower-hybrid gradient drift waves.

The problem of parametric and modulational instabilities of lower hybrid (LH) waves has a long history. The interest in them was motivated by such different problems as the problem of plasma heating and current drive generation in tokamaks and stellarators by the electromagnetic waves with the frequency near lower hybrid resonance,[14–16] by the problem of RF-discharges in low-pressure plasmas in the magnetic field,[17] by the problems of acceleration and heating of charged particles in space (ionospherical and magnetospherical) and astrophysical plasmas.[18–24]

In general, the modulational instability is sensitive to the wave dispersion. The dispersion of LH waves depends on the angle between the wavevector $\mathbf{k}$ and the external magnetic field. For LH waves propagating strictly perpendicular to the magnetic field, the dispersion is due to the thermal motion of plasma particles: either due to the electron Larmor radius effects (for dense plasma, $\omega_{pe} \gg \omega_{Be}$, where $\omega_{Be} = eB_0/m_e c$ is the electron cyclotron frequency, e is the proton charge, B_0 is the magnetic field strength, c is the speed of light, m_e is the electron mass, $\omega_{pe} = (4\pi e^2 n_0/m_e)^{1/2}$ is the electron plasma frequency, n_0 is the electron density) or due to the ion Debye length effects (for strong magnetic field, $\omega_{Be} \gg \omega_{pe}$) or the small electromagnetic correction to the LH frequency.[17,22,25–27] For oblique propagation of LH waves such that $(m_e/m_i)^{1/2} \le k_\parallel/k \ll 1$, the dispersion is due to the perturbed electron motion along the magnetic field.[14,22,28] Here m_i is the ion mass and $k_\parallel$ is the component of the wavevector along the equilibrium magnetic field.

To study the parametric and modulational instabilities of LH waves, both the fluid[14,22–27] and kinetic models[15,16,19,28,29] were used. A parametric decay of LH wave has been considered in a number of papers. The decay into another lower-frequency LH waves with $k_\parallel \neq 0$ in a cold homogeneous plasma was considered in Ref. 14; into the electromagnetic waves—in Ref. 30; into another LH wave and kinetic Alfvén wave—in Refs. 20–24, into the ion acoustic wave with formation of LH solitary structures—in Ref. 24; and into the LH wave and electron-drift mode in inhomogeneous plasma—in Ref. 16. The modulational instability of the LH waves has also been intensively studied. The process can be described as follows: a low-frequency quasi-mode (Ω, $\mathbf{q}$) with the phase velocity equal to the group velocity of pump LH wave couples with the primary (pump) wave (ω_0, $\mathbf{k}_0$) to produce two high-frequency LH sidebands ($\omega_0 \pm \Omega$, $\mathbf{k}_0 \pm \mathbf{q}$); these sidebands beat with the pump wave to produce a ponderomotive force driving low-frequency quasi-mode. The low-frequency quasi-mode is not a linear eigen-mode and is only driven nonlinearly. The modulational instability of LH waves propagating strictly perpendicular to the magnetic field has been studied in Refs. 17, 25–27. The case of obliquely propagating monochromatic LH waves has been considered in Refs. 28, 29, and 31–33 and modulational instability of the broad spectra of LH waves—in Ref. 34.

The equilibrium state of most Hall plasma devices is characterized by fast azimuthal rotation of electrons due to their $\mathbf{E}_0 \times \mathbf{B}_0$-drift and the presence of plasma density and magnetic field inhomogeneity in the (non-periodic) direction perpendicular to the magnetic field (the axial direction in case of the Hall thruster). In this paper, we show that in these conditions, the dispersion of the waves in the lower-hybrid frequency range propagating perpendicularly to the external magnetic field is due to the above noted gradients of the magnetic field and density. Here, we study the stability of such primary waves of finite amplitude (the pump wave) with respect to long-wavelength, low-frequency modulations.

The paper is organized as follows. The basic equations and dispersion properties of linear perturbations are discussed in Sec. II. The modulational instability of the lower-hybrid like primary wave is considered in Sec. III. The concluding remarks are presented in Sec. IV.

II. BASIC EQUATIONS AND LINEAR DISPERSION

We consider the perturbations in the local Cartesian coordinates (x, y, z) with the z coordinate along the equilibrium magnetic field, the y coordinate in the periodic azimuthal direction, and x coordinate in the direction of the inhomogeneity. Adopted for the coaxial Hall thrusters, in which the external magnetic field is assumed to be predominantly in the radial direction $\mathbf{B}_0 = B(x)\mathbf{e}_z + B_x(z)\mathbf{e}_x$, $B \gg B_x$, the local z coordinate is in the radial direction, the x coordinate is in the axial direction, and the y coordinate is in the symmetrical azimuthal direction. The equilibrium electric field $\mathbf{E}_0 = E_0\mathbf{e}_x$ is in the axial direction. For the cylindrical magnetrons (Penning discharge configuration) with axial magnetic field, the local z coordinate is axial, the x coordinate is radial, and the y coordinate is azimuthal, and the electric field is radial. We restrict ourselves to cold plasma approximation in the frame of the two-fluid plasma model.

We consider electrostatic perturbations, $\mathbf{E}' = -\nabla\phi$, propagating strictly perpendicular to the magnetic field. The electrons are assumed magnetized, $\omega \ll \omega_{Be}$ (ω is the oscillation frequency), and their motion across the equilibrium magnetic field is described by the standard expression

$$\mathbf{v}'_e = \frac{c}{B}\left[\frac{1}{B}\mathbf{B}_0 \times \nabla\phi + \frac{1}{\omega_{Be}} \times \left(\frac{\partial}{\partial t} + V_E\frac{\partial}{\partial y} + \frac{c}{B}(\mathbf{e}_z \times \nabla\phi)\cdot\nabla\right)\nabla\phi\right]. \quad (1)$$

Here $\phi = \phi(x, y)$ is the electrostatic potential of the perturbations, $V_E = -cE_0/B$ is the equilibrium azimuthal velocity of electrons. The first term corresponds to $\mathbf{E}' \times \mathbf{B}_0$-drift and the second is due to the electron inertial velocity across the external magnetic field. Substituting this expression in the electron continuity equation, we obtain the following equation for the electron density perturbation n'_e:

$$\left(\frac{\partial}{\partial t} + V_E\frac{\partial}{\partial y} + \frac{c}{B}(\mathbf{e}_z \times \nabla\phi)\cdot\nabla\right) \times \left(n'_e + \frac{cn_0}{B\omega_{Be}}\nabla^2\phi\right) - \frac{cn_0}{B}(\kappa_n - 2\kappa_B)\frac{\partial\phi}{\partial y} = 0, \quad (2)$$

where n_0 is the equilibrium plasma density, $\kappa_n = d\ln n_0/dx$, $\kappa_B = d\ln B/dx$. Plasma in the equilibrium state is assumed to be quasineutral. The term proportional to κ_B is due to compressibility of the electron $\mathbf{E}' \times \mathbf{B}_0$-drift related to the magnetic field gradient. The compressibility is calculated taking into account that the equilibrium plasma current in the low-pressure plasmas can be neglected, $\nabla \times \mathbf{B}_0 = 0$ (see the discussion in Ref. 10).

The ions are supposed to be unmagnetized, so that the ion dynamics is described by the equations

$$\frac{\partial n'_i}{\partial t} + \nabla\cdot\left((n_0 + n'_i)\mathbf{v}'_i\right) = 0, \quad (3)$$

$$\frac{\partial \mathbf{v}'_i}{\partial t} + (\mathbf{v}'_i\cdot\nabla)\mathbf{v}'_i = -\frac{e}{m_i}\nabla\phi, \quad (4)$$

where n'_i is the ion density perturbation and $\mathbf{v}'_i$ is the ion velocity perturbation.

To close the set of Equations (2)–(4), we use the Poisson equation

$$4\pi e(n'_e - n'_i) = \nabla^2\phi. \quad (5)$$

Linearization of the set of Equations (2)–(5) results in the dispersion relation

$$1 + \frac{\omega_{pe}^2}{\omega_{Be}^2} - \frac{\omega_{pi}^2}{\omega^2} - \frac{k_y}{k_\perp^2}\frac{\omega_{pe}^2(\kappa_n - 2\kappa_B)}{\omega_{Be}(\omega - \omega_E)} = 0, \quad (6)$$

where $\omega_E = k_yV_E$, $\omega_{pi} = (4\pi e^2 n_0/m_i)^{1/2}$ is the ion Langmuir frequency. In general, this dispersion relation describes the lower-hybrid gradient drift perturbations. The first two terms which are related to electron inertia and charge separation are important only for short-wavelength perturbations. In the long-wavelength limit, these terms are negligible. The dispersion relation in this limiting case has been studied, e.g., in Ref. 10 in the context of gradient drift instability. It can be strictly proven that the perturbations described by Eq. (6) are unstable if and only if the modified Simon-Hoh condition[35] is satisfied

$$\mu \equiv \mathbf{E}_0 \cdot \nabla\left(\frac{n_0}{B^2}\right) > 0. \tag{7}$$

In the frame of dispersion relation (6), this instability condition is valid for the perturbations of an arbitrary wavelength—for long-wavelength perturbations with the frequency $\omega \ll \omega_{lh}$ and for short-wavelength perturbations with $\omega \simeq \omega_{lh}$ (here $\omega_{lh} = \omega_{pi}\omega_{Be}/\sqrt{\omega_{pe}^2 + \omega_{Be}^2}$ is the lower-hybrid frequency). In the long-wavelength limit, this condition has been derived and discussed in detail in Ref. 10. We will be interested in the modulational instability of short-wavelength (high frequency) lower-hybrid gradient drift waves. Hereafter, we assume that $\mu < 0$ and the perturbations are *linearly stable*.

For typical Hall thrusters parameters with the equilibrium xenon plasma density $n_0 = 10^{12}\,\text{cm}^{-3}$ and the magnetic field $B = 100\,\text{G}$, the following estimates for the relevant frequencies apply:

$$\omega_{pe} = 5.6\times 10^{10}\,\text{s}^{-1}, \quad \omega_{pi} = 1.1\times 10^{8}\,\text{s}^{-1},$$
$$\omega_{Be} = 1.8\times 10^{9}\,\text{s}^{-1}, \quad \omega_{lh} = 3.5\times 10^{6}\,\text{s}^{-1}.$$

Typical values of the equilibrium electron azimuthal velocity are $V_E \sim 10^7\,\text{cm/s}$. It means that for channel radius $a = 4\,\text{cm}$ even for the lowest azimuthal mode with $m = 1$, we have $k_y = 1/a$ and $\omega_E \equiv k_y V_E \simeq \omega_{lh}$. For high-$m$, short-wavelength modes, the $\mathbf{E}\times\mathbf{B}$-drift frequency ω_E exceeds the lower-hybrid frequency, $\omega_E \gg \omega_{lh}$. Thus, both the eigenfrequency of the primary mode and the frequency of the long-wavelength quasi-mode are well below the ω_E frequency. Therefore, taking this into account, in application to the Hall thrusters, dispersion relation (6) can be simplified neglecting ω in combination with ω_E in the denominator of Eq. (6). Then dispersion relation (6) yields

$$\omega^2 = \omega_{lh}^2 \cdot \frac{k_\perp^2\lambda^2}{1 + k_\perp^2\lambda^2}. \tag{8}$$

Here λ is the characteristic space scale length

$$\lambda^2 \equiv \frac{\omega_{Bi} V_E}{(\kappa_n - 2\kappa_B)\omega_{lh}^2}. \tag{9}$$

In the short-wavelength limit, $k_\perp^2\lambda^2 \gg 1$, Equation (8) describes the lower-hybrid waves

$$\omega^2 = \omega_{lh}^2\left(1 - \frac{1}{k_\perp^2\lambda^2}\right). \tag{10}$$

Usually, the dispersion of lower-hybrid waves propagating strongly perpendicular to the magnetic field is related either with the electron Larmor radius (cold ions, dense plasma) or with the ion Debye length (hot ions, strong magnetic field). Here we see that their dispersion is due to plasma inhomogeneity and equilibrium azimuthal rotation of electrons.

In the opposite, long-wavelength limit, $k_\perp^2\lambda^2 \ll 1$, Equation (8) describes the lower-frequency mode

$$\omega^2 = \omega_{lh}^2 k_\perp^2\lambda^2 = \frac{k_\perp^2 \omega_{Bi} V_E}{\kappa_n - 2\kappa_B}. \tag{11}$$

In what follows, we will study the stability of a finite-amplitude short-wavelength lower-hybrid drift gradient mode with $k_\perp^2\lambda^2 \simeq 1$ with respect to long-wavelength low-frequency modulations, $q_\perp^2\lambda^2 \ll 1$, $\Omega \ll \omega_{lh}$.

First of all, we can significantly simplify the starting equations for further studies. Namely, under the above assumption $\omega \ll \omega_E$, it follows from Eq. (2) that the electron density perturbation can be described by the following expression:

$$n_e' = \frac{c n_0}{B\omega_{Be}}\left(\frac{(\kappa_n - 2\kappa_B)\omega_{Be}}{V_E}\phi - \nabla^2\phi\right). \tag{12}$$

Then, it follows from the Poisson equation (5) that the ion density perturbation takes the form

$$n_i' = \frac{1}{4\pi e}\left(1 + \frac{\omega_{pe}^2}{\omega_{Be}^2}\right)\left(-\nabla^2\phi + \frac{1}{\lambda^2}\phi\right). \tag{13}$$

Substituting this expression into Eq. (3), we obtain the following set of equations:

$$\frac{\partial}{\partial t}(\Phi - \nabla^2\Phi) + \nabla\cdot\mathbf{V} + \nabla\cdot((\Phi - \nabla^2\Phi)\mathbf{V}) = 0, \tag{14}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla\Phi. \tag{15}$$

Here time is normalized to ω_{lh}^{-1}, space variables x, y – to λ, and the dimensionless ion velocity $\mathbf{V}$ and electrostatic potential Φ are introduced as follows:

$$\Phi = \frac{e\phi}{m_i\omega_{lh}^2\lambda^2}, \quad \mathbf{V} = \frac{\mathbf{v}_i'}{\lambda\omega_{lh}}. \tag{16}$$

The set of Equations (14) and (15) will be used for studying the modulational instability of the lower-hybrid gradient drift wave.

III. MODULATIONAL INSTABILITY OF LOWER-HYBRID GRADIENT DRIFT WAVE

We assume the primary lower-hybrid gradient drift wave in the form

$$(\Phi^{(0)}, \mathbf{V}^{(0)}) = (\Phi_0, \mathbf{V}_0)\exp(-i\omega_0 t + i\mathbf{k}_0\mathbf{x}) + (\Phi_0^\star, \mathbf{V}_0^\star)\exp(i\omega_0 t - i\mathbf{k}_0\mathbf{x}), \tag{17}$$

where $f^\star$ implies a complex conjugate to f, ω_0 is the wave frequency normalized to ω_{lh}, $\mathbf{k}_0$ is the wavevector normalized to $1/\lambda$, and

$$\omega_0 \equiv \omega(\mathbf{k}_0) = \frac{k_0}{\sqrt{1 + k_0^2}}, \quad \mathbf{V}_0 = \frac{\mathbf{k}_0}{\omega_0}\Phi_0. \tag{18}$$

We consider the long-wavelength, lower-frequency modulation of the form

$$(\Psi, \mathbf{U}) = (\Psi_q, \mathbf{U}_q)\exp(-i\Omega t + i\mathbf{q}\mathbf{x}), \qquad (19)$$

where Ω and $\mathbf{q}$ are dimensionless frequency and wavevector of the modulation normalized to ω_{lh} and to $1/\lambda$, correspondingly. It is assumed that $\Omega \ll 1$ and $q \ll 1$. Due to nonlinear effects, the pump wave and long-wavelength perturbation (quasi-mode) are coupled to two sideband lower-hybrid gradient drift waves

$$(\Phi^{(s)}, \mathbf{V}^{(s)}) = (\Phi_+, \mathbf{V}_+)\exp(-i\omega_+ t + i\mathbf{k}_+\mathbf{x}) + (\Phi_-, \mathbf{V}_-)\exp(i\omega_- t - i\mathbf{k}_-\mathbf{x}). \qquad (20)$$

Here, $\omega_\pm = \omega_0 \pm \Omega$, $\mathbf{k}_\pm = \mathbf{k}_0 \pm \mathbf{q}$. One can easily check that three-wave resonance

$$\omega(\mathbf{k}_\pm) = \omega(\mathbf{k}_0) \pm \omega(\mathbf{q}), \qquad (21)$$

for eigenmodes with $\omega(\mathbf{q}) = q$ cannot be satisfied and only modulational instability with excitation of quasi-mode with the phase velocity equal to the group velocity $\mathbf{v}_g = \partial\omega_0/\partial\mathbf{k}_0$ of the pump lower-hybrid gradient drift wave $\Omega = \mathbf{q}\mathbf{v}_g$ is possible.

Let us consider the nonlinear interaction of the pump wave, its long-wavelength modulation, and two sidebands. We substitute

$$\Phi = \Phi^{(0)} + \Phi^{(s)} + \Psi, \quad \mathbf{V} = \mathbf{V}^{(0)} + \mathbf{V}^{(s)} + \mathbf{U}$$

in Eqs. (14) and (15), linearize the equations with respect to the amplitude of the modulation and sideband waves, and separate different spatio-temporal harmonics of the perturbation.

Then two sideband waves are described by the equations

$$\begin{aligned}
&\omega_+\left(1 + k_+^2\right)\Phi_+ - (\mathbf{k}_+\mathbf{V}_+) \\
&\quad = \Phi_0\left[\left(1 + k_0^2\right)(\mathbf{k}_+\mathbf{U}_q) + \frac{\mathbf{k}_0\mathbf{k}_+}{\omega_0}\Psi_q\right], \\
&\omega_+(\mathbf{k}_+\mathbf{V}_+) - k_+^2\Phi_+ \\
&\quad = \frac{\Phi_0}{\omega_0}\left[(\mathbf{k}_0\mathbf{k}_+)(\mathbf{k}_0\mathbf{U}_q) + (\mathbf{k}_0\mathbf{q})(\mathbf{k}_+\mathbf{U}_q)\right],
\end{aligned} \qquad (22)$$

and

$$\begin{aligned}
&\omega_-\left(1 + k_-^2\right)\Phi_- - (\mathbf{k}_-\mathbf{V}_-) \\
&\quad = \Phi_0^\star\left[\left(1 + k_0^2\right)(\mathbf{k}_-\mathbf{U}_q) + \frac{\mathbf{k}_0\mathbf{k}_-}{\omega_0}\Psi_q\right], \\
&\omega_-(\mathbf{k}_-\mathbf{V}_-) - k_-^2\Phi_- \\
&\quad = \frac{\Phi_0^\star}{\omega_0}\left[(\mathbf{k}_0\mathbf{k}_-)(\mathbf{k}_0\mathbf{U}_q) - (\mathbf{k}_0\mathbf{q})(\mathbf{k}_-\mathbf{U}_q)\right].
\end{aligned} \qquad (23)$$

The equations for the long-wavelength modulation are

$$\begin{aligned}
\Omega\Psi_q - (\mathbf{q}\mathbf{U}_q) &= (1 + k_0^2)[\Phi_0(\mathbf{q}\mathbf{V}_-) + \Phi_0^\star(\mathbf{q}\mathbf{V}_+)] \\
&\quad + (1 + k_+^2)\Phi_+(\mathbf{q}\mathbf{V}_0^\star) + (1 + k_-^2)\Phi_-(\mathbf{q}\mathbf{V}_0), \\
\Omega\mathbf{U}_q - \mathbf{q}\Psi_q &= (\mathbf{k}_0\mathbf{V}_-)\mathbf{V}_0 + (\mathbf{k}_+\mathbf{V}_0^\star)\mathbf{V}_+ \\
&\quad - (\mathbf{k}_0\mathbf{V}_+)\mathbf{V}_0^\star - (\mathbf{k}_-\mathbf{V}_0)\mathbf{V}_-.
\end{aligned} \qquad (24)$$

Now we solve Eqs. (22) and (23) for Φ_+ and for Φ_-, correspondingly. Then with the required accuracy, we obtain

$$\begin{aligned}
\Phi_+ &= \frac{k_0^2\Phi_0}{\omega_0 D_+}(\omega_0\Psi_q + 2(\mathbf{k}_0\mathbf{U}_q)), \\
\Phi_- &= \frac{k_0^2\Phi_0^\star}{\omega_0 D_-}(\omega_0\Psi_q + 2(\mathbf{k}_0\mathbf{U}_q)),
\end{aligned} \qquad (25)$$

where

$$D_\pm = (1 + k_\pm^2)(\omega_0 \pm \Omega)^2 - k_\pm^2. \qquad (26)$$

Solving Eqs. (22) and (23), we kept only the main order terms on the right hand sides and neglected the terms which are small as q/k_0 and Ω/ω_0.

We have already mentioned that the decay instability is impossible and therefore we will study the excitation of non-eigenmodes with $\Omega \approx \mathbf{q}\mathbf{v}_g$. Taking this into account, we simplify the expressions defining $D_\pm$ near $\Omega = \mathbf{q}\mathbf{v}_g$ and obtain

$$D_\pm = 2\omega_0\left(1 + k_0^2\right)\left[\pm\left(\Omega - \mathbf{q}\frac{\partial\omega_0}{\partial\mathbf{k}_0}\right) + \Delta\right], \qquad (27)$$

where

$$\begin{aligned}
\mathbf{v}_g &= \frac{\partial\omega_0}{\partial\mathbf{k}_0} \equiv \frac{\mathbf{k}_0}{k_0\left(1 + k_0^2\right)^{3/2}}, \\
\Delta &\equiv \frac{\left(1 + 4k_0^2\right)q^2}{2k_0\left(1 + k_0^2\right)^{5/2}}\left\{\cos^2\theta - \frac{1 + k_0^2}{1 + 4k_0^2}\right\}.
\end{aligned} \qquad (28)$$

Here $\cos\theta = \mathbf{k}_0\mathbf{q}/k_0 q$.

The relation between the sidebands of the ion velocity and potential can be simplified by neglecting the nonlinear terms

$$\mathbf{V}_\pm = \frac{\mathbf{k}_\pm}{\omega_\pm}\Phi_\pm.$$

These relations are used in nonlinear terms on the right hand sides of Equation (24). As above, we keep only the nonlinear terms of the main order and neglect the terms of the order q/k_0 and Ω/ω_0. Then Eq. (24) takes the form

$$\begin{aligned}
\Omega\Psi_q - (\mathbf{q}\mathbf{U}_q) &= \frac{2(\mathbf{k}_0\mathbf{q})\left(1 + k_0^2\right)}{\omega_0}\left(\Phi_0^\star\Phi_+ + \Phi_0\Phi_-\right), \\
\Omega\mathbf{U}_q - \mathbf{q}\Psi_q &= \frac{k_0^2}{\omega_0^2}\mathbf{q}\left(\Phi_0^\star\Phi_+ + \Phi_0\Phi_-\right).
\end{aligned} \qquad (29)$$

Solving these equations, we find that

$$\mathbf{k}_0\mathbf{U}_q = \frac{\mathbf{k}_0\mathbf{q}}{q^2}\mathbf{q}\mathbf{U}_q, \;\; \mathbf{q}\mathbf{U}_q = q^2\frac{\Omega k_0^2 + 2\omega_0(\mathbf{k}_0\mathbf{q})\left(1 + k_0^2\right)}{2\omega_0\Omega(\mathbf{k}_0\mathbf{q})\left(1 + k_0^2\right) + q^2k_0^2}\Psi_q. \qquad (30)$$

Finally, we multiply the equation for velocity $\mathbf{U}_q$ in (29) by $\mathbf{q}$. and substitute expressions (25) and (27)–(30) in the resulting equation and in the equation for Ψ_q in (29). Then we finally arrive at the dispersion relation for long-wavelength, low-frequency quasi-mode

$$\left(\Omega-\mathbf{q}\mathbf{v}_g\right)^2-\Delta^2=k_0\left(1+k_0^2\right)^{5/2}\Delta \cdot\frac{1+k_0^2+4\cos^2\theta\left[1+\left(1+k_0^2\right)^2\right]}{\left(1+k_0^2\right)^3-\cos^2\theta}\cdot|\Phi_0|^2. \tag{31}$$

It follows from this equation that the modulational instability, $\Gamma=\mathrm{Im}\ \Omega>0$, can take place only when $\Delta<0$, i.e., according to Eq. (28), only if the condition

$$\cos^2\theta<\frac{1+k_0^2}{1+4k_0^2} \tag{32}$$

is satisfied. Since

$$\Delta\equiv-\frac{1}{2}\,q_iq_j\cdot\frac{\partial^2\omega_0}{\partial k_{0i}\partial k_{0j}},$$

the necessary condition of instability $\Delta<0$ is equivalent to the well-known Lighthill criterion generalized in application to the 2D-problem. The necessary and sufficient instability condition for the modulational instability defines the amplitude of the primary wave that can become unstable. This condition takes the form

$$|\Phi_0|^2>\frac{|\Delta|}{k_0\left(1+k_0^2\right)^{5/2}}\cdot\frac{\left(1+k_0^2\right)^3-\cos^2\theta}{1+k_0^2+4\cos^2\theta\left[1+\left(1+k_0^2\right)^2\right]}. \tag{33}$$

Note that the dispersion relation has to be modified for the perturbations with such $\mathbf{q}$ that $\Delta\approx0$. In this case, it is necessary to take into account the terms of the order q^4 in $D_\pm$.

The phase velocity of the quasi-mode is exactly equal to the group velocity of the primary lower-hybrid gradient drift mode, so that the real part of its frequency is: $\mathrm{Re}\,\Omega=\mathbf{q}\mathbf{v}_g$. Returning to dimensional frequency and wavevectors and using Eq. (28), we find that the real part of frequency of the unstable low-frequency quasimode is

$$\mathrm{Re}\,\Omega=\omega_{lh}\cdot\frac{q\lambda\cos\theta}{\left(1+k_0^2\lambda^2\right)^{3/2}}. \tag{34}$$

It is worthy to notice that according to Eq. (34) for the pump wave with $k_0\lambda\leq1$, the real part of the quasimode frequency is of the order

$$\mathrm{Re}\,\Omega\simeq\omega_{lh}\cdot(q\lambda)=q\left(\frac{eE_0}{m_i(2\kappa_B-\kappa_n)}\right)^{1/2}. \tag{35}$$

This frequency is much smaller than ω_{lh} and *weakly depends* on the amplitude of magnetic field. In some aspects—frequency range and weak dependence on the magnetic field strength—these properties remind the properties of spokes observed in Hall thrusters,[2,13] and whose origin is still poorly explained. One can speculate that nonlinear effects like modulational instability described above could be responsible for spokes in Hall thrusters.

Instability condition (33) in dimensional variables takes the form

$$|\phi_0|^2>\frac{q^2}{2k_0^2}\cdot\frac{1+4k_0^2\lambda^2}{\left(1+k_0^2\lambda^2\right)^5}\cdot\left(\frac{1+k_0^2\lambda^2}{1+4k_0^2\lambda^2}-\cos^2\theta\right)\cdot\frac{\left(1+k_0^2\lambda^2\right)^3-\cos^2\theta}{1+k_0^2\lambda^2+4\cos^2\theta\left[1+\left(1+k_0^2\lambda^2\right)^2\right]}\cdot\left(\frac{E_0}{2\kappa_B-\kappa_n}\right)^2, \tag{36}$$

where ϕ_0 is the electrostatic potential of the pump wave. For the pump wave with $k_0\lambda\leq1$, we find the estimate for the modulational instability threshold

$$|\phi_0|\simeq\frac{q}{k_0}\cdot\frac{E_0}{2\kappa_B-\kappa_n}. \tag{37}$$

On the other hand, for shorter-length pump wave with $k_0\lambda\gg1$, the real part of quasimode frequency is much lower and is of the order

$$\mathrm{Re}\ \Omega\simeq\omega_{lh}\cdot\frac{q}{k_0^3\lambda^2}=\omega_{lh}^3\cdot\frac{q}{k_0^3}\cdot\frac{2\kappa_B-\kappa_n}{eE_0/m_i}. \tag{38}$$

In dense plasmas such that $\omega_{pe}^2\gg\omega_{Be}^2$, this frequency scales as B^3.

It follows from dispersion relation (31) that the growth rate of the low-frequency quasimode is described by the equation

$$\Gamma^2=\omega_{lh}^2\cdot q^2\lambda^2\,\Delta_0(\alpha|\Phi_0|^2-q^2\lambda^2\,\Delta_0). \tag{39}$$

Here

$$\alpha=k_0\lambda\left(1+k_0^2\lambda^2\right)^{5/2}\cdot\frac{1+k_0^2\lambda^2+4\cos^2\theta\left[1+\left(1+k_0^2\lambda^2\right)^2\right]}{\left(1+k_0^2\lambda^2\right)^3-\cos^2\theta},$$

$$\Delta_0\equiv\frac{\left(1+4k_0^2\lambda^2\right)}{2k_0\lambda\left(1+k_0^2\lambda^2\right)^{5/2}}\left\{\frac{1+k_0^2\lambda^2}{1+4k_0^2\lambda^2}-\cos^2\theta\right\}. \tag{40}$$

The growth rate has its maximum of

$$\Gamma_{max}=\frac{1}{2}\alpha\left(\frac{2\kappa_B-\kappa_n}{E_0}\right)^2|\phi_0|^2\,\omega_{lh}, \tag{41}$$

for the wavevector of the order of

$$q_{max}^2=\frac{\alpha}{2\lambda^2\Delta_0}\left(\frac{2\kappa_B-\kappa_n}{E_0}\right)^2|\phi_0|^2. \tag{42}$$

For the pump wave with $k_0\lambda\leq1$, the most unstable quasimode is characterized by the following growth rate and wavevector:

$$\Gamma_{max} \simeq k_0\lambda \cdot \left(\frac{2\kappa_B - \kappa_n}{E_0}\right)^2 |\phi_0|^2 \, \omega_{lh},$$
$$q_{max} \simeq k_0 \cdot \frac{2\kappa_B - \kappa_n}{E_0} |\phi_0|. \tag{43}$$

The growth rate of modulational instability has maximum when the wavevector of perturbation is perpendicular to the wavevector of the pump wave.

IV. CONCLUSION

Under typical conditions of Hall plasma devices, we have considered the electrostatic perturbations propagating perpendicular to the magnetic field with the frequencies in the range $\omega_{Bi} < \omega \leq \omega_{lh}$. We have derived a simplified set of nonlinear equations (14) and (15) describing the electrostatic perturbations in the above frequency range. We have used these equations to study the modulational instability of lower-hybrid gradient drift wave. We have found an instability criterion described by inequalities (32) and (33). For the primary lower-hybrid gradient drift waves with the wave-number $k_0\lambda \leq 1$, the frequency of unstable quasimode given by Eq. (35) is much lower than the frequency related to azimuthal electron $\mathbf{E}_0 \times \mathbf{B}_0$ drift, which is consistent with the frequency range of typically observed low frequency rotating structures. It is also interesting that the unstable quasimode frequency only slightly depends on the strength of external magnetic field. These properties of unstable quasimode suggest that the low-frequency large scale structures often observed in Hall plasma devices, such as thrusters[37] and magnetrons,[38,39] may be explained as a result of nonlinear condensation (modulational instability) of small-scale fluctuations of the lower-hybrid type. The background of primary lower-hydrid waves can be directly excited by small scale instabilities, such as that due to collisions of electrons with neutral atoms,[11] shear flows,[36] or small scale electron cyclotron $\mathbf{E} \times \mathbf{B}$ modes.[1] In this study, the primary wave was assumed to be neutrally stable. Such stationary state can be achieved as a result of saturation due to nolinear interaction of primary modes.

We should note that similar to Ref. 15, we consider the simplest case of the modulational instability neglecting the effect of inhomogeneity of plasma parameters. In fact, the inhomogeneous magnetic field as well as the inhomogeneous electric field itself may result in the shear of the $\mathbf{E} \times \mathbf{B}$ flow that affects both the linear stability of primary modes[36,40] and the nonlinear modulational instability. Shearing of the $\mathbf{E} \times \mathbf{B}$ flow[40] may also prevent the mode growth maintaining its amplitude below the threshold for the instability. The effect of the inhomogeneous profiles of $\mathbf{E} \times \mathbf{B}$ velocity on the spectral stability of global (nonlocal) low-hybrid modes was recently studied in Ref. 41, while the explicit shear flow effects on standard drift waves were studied with the non-modal approach in Ref. 40. While, in general, the full nonlinear theory has to include all these effects, such analysis, however, is outside the scope of our paper. For Hall thruster configuration, in the central region of the ionization zone, the electric field decay is accompanied by the decay of the magnetic field, thus making the $\mathbf{E} \times \mathbf{B}$ velocity more uniform. It was shown in Ref. 41 that in the short wavelength regime, unstable low-hybrid modes destabilized by density and magnetic field gradients are clustered into the local wave-packets. It is also interesting to note that the structure of the equations for gradient-drift modes in Eq. (6) is somewhat analogous to the equations for the diocotron mode. Therefore, one can expect that some results of the non-modal theory of the diocotron mode[42] can be applicable to the stability of the lower-hybrid gradient drift modes.

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