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IONOSPHERIC AND SPACE PLASMAS

Nonlinear Equation for Farley–Buneman Waves in Multispecies Plasma[1]

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Abstract—The nonlinear equation describing the Farley–Buneman (FB) waves in multispecies collisional plasmas is derived by employing the multiple-scale reduction analysis. It is shown that the presence of several ion species with different collisionalities and different ion masses removes the degeneracy of the nonlinear equation and generates the nonlinear terms resulting in wave steepening and wave breaking. This effect may be responsible for formation of one-dimensional coherent FB waves of a finite amplitude.

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1. INTRODUCTION

The Farley–Buneman (FB) instability [1–5] is typically considered as a source of electrojet irregularities observed in the E-region of Earth's ionosphere at a height where the electrons are strongly magnetized ($\omega_{ce} \gg \nu_{en}$) and ions are unmagnetized ($\omega_{ci} \ll \nu_{in}$), where $\omega_{c(e,\ i)}$ is the gyrofrequency and $\nu_{(e,\ i)n}$ is the collision frequency of charged species with the neutrals. It has been extensively studied in linear theory (see, e.g., [4–9] and references therein). There remains, however, a number of important observational features that need to be explained on the basis of nonlinear theory, such as the saturation of the instability, nature of the nonlinear states and the resulting spectral characteristics of developed turbulence.

The nonlinear theory of the FB instability has been investigated for the mode saturation as a result of nonlinear electron drift and diffusion, as well as nonlinear energy flow from linearly excited modes to the damped modes [5, 10–15]. Saturation mechanism of FB modes via the nonlinear wave coupling has been studied through computer simulations in [15, 16]. The simulations [17] have shown that the mode saturation depends on flow angle, that is the angle between the directions of wave propagation and the drift velocity. Further, it was demonstrated [14] that the excitation of small scale secondary waves plays an important role in the nonlinear saturation of the instability. These saturation mechanisms are related to the nonlinearity due to the electron $\mathbf{E} \times \mathbf{B}$ drift, which is essentially a two-dimensional effect in a plane perpendicular to the magnetic field direction. Respective nonlinear equation with two-dimensional (vector) nonlinearity was derived, e.g., in [18]. This nonlinearity was found to be responsible for the energy transfer between small (FB) and large scale structures resulting in the generation of secondary modes at finite flow angles [19, 20]. It was also shown that this mechanism can stabilize FB instability by coupling of an unstable mode with two damped modes [9]. Nonlinear interaction of FB mode with external RF fields was also considered and suggested as a stabilization mechanism from intense radar beams [21].

The two-dimensional nonlinearity is likely to be responsible for often observed finite flow angle modes and for the mode spreading into the linearly stable regions [20]. However, some observations indicate the presence of large amplitude square waves [22, 23] which resemble coherent one-dimensional waves. The mechanism for formation of such structures (most likely to be one-dimensional) is not clear. It was suggested [24] that such square wave structures may be explained by wave breaking of large amplitude waves similar to the breaking of the water waves.

Wave breaking is a result of nonlinear dependence of the wave phase velocity on the wave amplitude.

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Within the approximation of weakly nonlinear waves, the model nonlinear equation for one dimensional wave demonstrating the wave breaking phenomena can be written in the form:

$$\frac{\partial f}{\partial \tau} + f\frac{\partial f}{\partial \xi} - D\frac{\partial^2 f}{\partial \xi^2} + \rho^2\frac{\partial^3 f}{\partial \xi^3} = 0. \quad (1)$$

The second term in Eq. (1) is a nonlinear term which leads to the wave breaking, the third term is the diffusion/dissipation with D as a diffusion coefficient, and the fourth term represents dispersion with ρ^2 as a dispersion parameter. The FB waves are almost dispersion-free (at least in the meter-scale wavelength region) [20]. In absence of dispersion, $\rho^2 = 0$, Eq. (1) becomes Burgers equation, which was derived for many type of water waves and waves in plasma. For unstable FB mode, the diffusion term is, in fact, negative with $D < 0$, resulting in growing modes as $f \sim \exp(-i\omega\tau + ik\xi)$ with frequency $\omega = ik^2D$. Unstable waves can grow to large amplitude when the nonlinear term becomes significant leading to the wave steepening and breaking [25]. One-dimensional (scalar) nonlinearity is the main nonlinear effect for sound type waves (such as ion sound), where it is crucial for the formation of solitons, periodic nonlinear waves (cnoidal waves), and shock waves [25].

The FB instability is often thought as a special case of destabilized ion-sound waves. However, FB waves possess unique degeneration due to remarkable cancelation of the first-order (quadratic) nonlinearities of the sound wave type. It was shown [18, 26–28] that the first order nonlinear terms are exactly canceled in the regime $\omega < \nu_i$, typically considered for ionosphere application. This property precludes the wave steepening and wave breaking for FB waves. In this work, we revisit the problem of the nonlinear equation for one-dimensional FB waves. By considering weakly nonlinear waves and employing the technique of reductive perturbation method, we derive a nonlinear equation for FB waves. We show that the above noted degeneracy is removed and one-dimensional convective nonlinearity is retained in multispecies plasmas resulting in wave steepening and wave breaking. In addition, it is shown that there exists one-dimensional nonlinear diffusion nonlinearity related to the ion inertia [29]. The normalized nonlinear equation for one dimensional FB waves in a multispecies plasma can be written in the form

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial \xi} + (D_I - D_D)\frac{\partial^3 U}{\partial \xi^2} - D_{\rm NL}\frac{\partial^2 U^2}{\partial \xi^2} = 0. \quad (2)$$

The linear diffusion-type coefficients, $D_I > 0$ and $D_D > 0$, are responsible for the instability and stabilizing diffusion, respectively, and $D_{\rm NL}$ describes nonlinear diffusion effects. The derivation of this equation is a subject of this manuscript.

2. BASIC EQUATIONS

Our model is based on the basic fluid equations for ions and electrons. We consider the magnetic field along the z direction, $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, and the electric field along the y direction, $\mathbf{E} = E\hat{\mathbf{y}}$. This will thus give the $\mathbf{E} \times \mathbf{B}$ drift along the x direction. Ions are assumed to be isothermal (T_i is constant) and unmagnetized as the ion cyclotron frequency is much smaller than the ion-neutral collision frequency ($\omega_{ci} = eB_0/m_ic \ll \nu_{in}$). The ion dynamics can be described by continuity and momentum equations as

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i\mathbf{v}_i) = 0, \quad (3)$$

$$m_i\frac{d\mathbf{v}_i}{dt} = e\mathbf{E} - \frac{\nabla p_i}{n_i} - m_i\nu_{in}\mathbf{v}_i, \quad (4)$$

where the variables have their standard meanings. The electric field is given by $\mathbf{E} = -\nabla\phi$, with ϕ as the electrostatic potential, and the pressure gradient is $\nabla p_i = T_i\nabla n_i$. Assuming that the ion velocity can be expressed in terms of a velocity potential as $\mathbf{v}_i = -\nabla\chi$, the above equations for ions can be written as

$$\frac{\partial n_i}{\partial t} - \nabla \cdot (n_i\nabla\chi) = 0, \quad (5)$$

$$\left(\frac{\partial}{\partial t} + \nu_{in}\right)\nabla^2\chi = \nabla^2\left(\frac{e}{m_i}\phi + \frac{T_i}{m_i}\ln\frac{n_i}{n_0} + \frac{(\nabla\chi)^2}{2}\right). \quad (6)$$

The electrons are considered as isothermal (T_e is constant) and magnetized as their cyclotron frequency is much greater than the electron–neutral collision frequency ($\omega_{ce} = eB_0/m_ec \gg \nu_{en}$) and their motion is dominated by the magnetic field. The electron dynamics is also described by the continuity and momentum equations as:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\mathbf{v}_e) = 0, \quad (7)$$

$$m_e\frac{d\mathbf{v}_e}{dt} = -e\left[\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B}\right] - \frac{\nabla p_e}{n_e} - m_e\nu_{en}\mathbf{v}_e, \quad (8)$$

with $\mathbf{E} = -\nabla\phi$ and the pressure gradient as $\nabla p_e = T_e\nabla n_e$. Electron inertia can be neglected since $\omega < \omega_{ce}$ and $\nu_{en} > \omega$, so the electron collisional response is more important than the inertial (low hybrid mode thus is not considered). Thus, neglecting the left-hand side in Eq. (8), the momentum equation can be solved to obtain the electron perpendicular velocity as

$$\mathbf{v}_e = \mathbf{V}_E + \mathbf{V}_{pe} + \mathbf{V}_\nu, \quad (9)$$

where $\mathbf{V}_E$ is the $\mathbf{E} \times \mathbf{B}$ drift, $\mathbf{V}_{pe}$ is the diamagnetic drift, and $\mathbf{V}_\nu$ is the next-order drift given by

$$\mathbf{V}_E = \frac{c}{B_0}\mathbf{b} \times \nabla\phi, \tag{10}$$

$$\mathbf{V}_{pe} = -\frac{c}{en_eB_0}\mathbf{b} \times \nabla p_e, \tag{11}$$

$$\begin{aligned}\mathbf{V}_\nu &= -\frac{\nu_{en}}{\omega_{ce}}\mathbf{b} \times (\mathbf{V}_E + \mathbf{V}_{pe}) \\ &= \frac{\nu_{en}}{\omega_{ce}}\frac{c}{B_0}\left(\nabla\phi - \frac{T_e}{en_e}\nabla n_e\right),\end{aligned} \tag{12}$$

with $\omega_{ce} > 0$. Note that we consider the modes which have no variations along the magnetic field ($k_z = 0$), so the electron parallel velocity (along the magnetic field) can be neglected here. Now, using Eq. (9) in electron continuity equation (7), we obtain

$$\begin{aligned}&\frac{\partial n_e}{\partial t} + \mathbf{V}_E \cdot \nabla n_e + \frac{\nu_{en}c}{B_0\omega_{ce}}\nabla \cdot (n_e\nabla\phi) \\ &\quad - \frac{\nu_{en}cT_e}{eB_0\omega_{ce}}\nabla^2 n_e = 0.\end{aligned} \tag{13}$$

Equations (5), (6), and (13) together with the quasi-neutrality ($n_e \approx n_i = n$) condition make up our basic model.

3. DERIVATION OF THE NONLINEAR EVOLUTION EQUATION BY THE REDUCTIVE PERTURBATION THEORY

The reductive perturbation theory, widely used in nonlinear optics and wave dynamics [30–32], allows to reduce a complex system of nonlinear equations (5), (6), and (13) to a single nonlinear equation. The perturbative expansion is performed in a small parameter corresponding to the wave amplitude. In our case, the amplitude expansion is combined with the expansion near the marginal stability. In the leading order, one has the stable propagating wave; effects of the nonlinearity and instability are included in the next order. Thus, the reduced (simplified) nonlinear equation includes the dominant nonlinear effects and diffusive (negative or positive) terms near the marginal stability.

Technically, the expansion is done by introducing the stretching coordinates, such as the spatial variable $\xi = x - ut$ and slow time dependence variable $\tau = t$. All variables are represented in the form $X = X(\tau, \xi)$, where $X = (\tilde{n}, \tilde{\chi}, \tilde{\phi})$ represents the perturbed quantities. The parameter u corresponds to the phase velocity of the stable linear wave and the dependence on the τ and ξ variables is due to the weak nonlinear terms and weak deviation from marginal stability.

First, we review the derivation of the nonlinear equation for the single ion species case. All variables are considered in a form of the series expansion

$$\tilde{X} = \tilde{X}^{(1)} + \tilde{X}^{(2)} + \ldots. \tag{14}$$

Using stretching coordinates along with the expansion of dependent variables, from Eqs. (5), (6), and (13), the lowest order of equations are

$$-u\partial_\xi\tilde{n}^{(1)} - n_0\nabla^2\tilde{\chi}^{(1)} = 0, \tag{15}$$

$$\nu_i\tilde{\chi}^{(1)} = \frac{e}{m_i}\tilde{\phi}^{(1)}, \tag{16}$$

$$(-u + V_0)\partial_\xi\tilde{n}^{(1)} + \frac{\nu_{en}cn_0}{B_0\omega_{ce}}\nabla_\perp^{2\,(1)} = 0, \tag{17}$$

where V_0 is the equilibrium $\mathbf{E} \times \mathbf{B}$ drift in the x direction. These equations form the homogenous system for three first-order variables $\tilde{n}^{(1)}$, $\tilde{\chi}^{(1)}$, and $\tilde{\phi}^{(1)}$. The solvability condition for this system defines the phase velocity for the linear wave,

$$-1 + \frac{V_0}{u} = \psi, \tag{18}$$

with $\psi = \nu_{en}\nu_{in}/\omega_{ce}\omega_{ci}$. In the second order, one takes into account the nonlinear terms and the terms responsible for the instability. These equations are

$$\begin{aligned}&\frac{\partial}{\partial\tau}\tilde{n}^{(1)} - u\partial_\xi\tilde{n}^{(2)} - n_0\nabla^2\tilde{\chi}^{(2)} \\ &\quad - \nabla \cdot (\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)}) = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}\nu_{in}\tilde{\chi}^{(2)} - u\partial_\xi\tilde{\chi}^{(1)} &= \frac{e}{m_i}\tilde{\phi}^{(2)} \\ &+ \frac{T_i}{m_i}\frac{\tilde{n}^{(1)}}{n_0} + \frac{1}{2}(\nabla\tilde{\chi}^{(1)})^2,\end{aligned} \tag{20}$$

$$\begin{aligned}&\frac{\partial}{\partial\tau}\tilde{n}^{(1)} + (-u + V_0)\partial_\xi\tilde{n}^{(2)} + \frac{\nu_{en}cn_0}{B_0\omega_{ce}}\nabla_\perp^2\tilde{\phi}^{(2)} \\ &+ \frac{\nu_{en}c}{B_0\omega_{ce}}\nabla_\perp \cdot (\tilde{n}^{(1)}\nabla_\perp\tilde{\phi}^{(1)}) - \frac{\nu_{en}cT_e}{eB_0\omega_{ce}}\nabla_\perp^2\tilde{n}^{(1)} \\ &+ \frac{c}{B_0}\mathbf{b} \cdot \nabla\tilde{\phi}^{(1)} \times \nabla\tilde{n}^{(1)} = 0.\end{aligned} \tag{21}$$

The secular terms (related to the linear terms $\tilde{n}^{(2)}$, $\tilde{\phi}^{(2)}$, and $\tilde{\chi}^{(2)}$) are removed from Eqs. (19)–(21) by the condition in Eq. (18) and the remaining equation gives the evolution equation for $\tilde{n}^{(1)}$, which is defined by the nonlinear effects and weak deviation from marginal stability. Equations (19)–(21) have three different types of nonlinear terms: the two-dimensional vector nonlinearity $(c/B_0)\mathbf{b} \cdot \nabla\tilde{\phi}^{(1)} \times \nabla\tilde{n}^{(1)}$, which originates from $\mathbf{E} \times \mathbf{B}$ convection in the electron continuity equation; the convective nonlinearity $(\nabla\tilde{\chi}^{(1)})^2/2$, which originates from $\mathbf{v} \cdot \nabla\mathbf{v}$ terms in the equation of ion motion; and another convective type nonlinearities in the ion $\nabla \cdot (\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)})$ and the electron

$(\nu_{en}c/B_0\omega_{ce})\nabla_\perp\cdot(\tilde{n}^{(1)}\nabla_\perp\tilde{\phi}^{(1)})$ continuity equations. The $\mathbf{v}\cdot\nabla\mathbf{v}$ nonlinear term, or in another form $(\nabla\tilde{\chi}^{(1)})^2$, corresponds to the nonlinear diffusion. The convective terms, $\nabla\cdot(\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)})$ and $\nabla_\perp\cdot(\tilde{n}^{(1)}\nabla_\perp\tilde{\phi}^{(1)})$ are present in one-dimensional case and normally would be responsible for the wave steepening and breaking. There is, however, remarkable exact cancelation between these terms and the lowest order scalar nonlinearity does not appear in nonlinear equation [28].

This cancelation is demonstrated as follows. Excluding $\tilde{\chi}^{(2)}$ from Eqs. (19) and (20), one obtains

$$\begin{aligned}\partial_\xi\tilde{n}^{(2)} &= \frac{1}{u}\frac{\partial}{\partial\tau}\tilde{n}^{(1)}\\ &-\frac{n_0}{u}\nabla^2\left(\frac{u}{\nu_{in}}\partial_\xi\tilde{\chi}^{(1)}+\frac{e}{m_i\nu_{in}}\tilde{\phi}^{(2)}\right.\\ &+\left.\frac{T_i}{m_i\nu_{in}}\frac{\tilde{n}^{(1)}}{n_0}+\frac{1}{2\nu_{in}}(\nabla\tilde{\chi}^{(1)})^2\right)\\ &-\frac{1}{u}\nabla\cdot(\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)}).\end{aligned} \tag{22}$$

The final nonlinear equation is obtained by excluding $\tilde{n}^{(2)}$ between Eqs. (21) and (22). The nonlinear terms in these equations, $\nabla\cdot(\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)})$ and $\nabla_\perp\cdot(\tilde{n}^{(1)}\nabla_\perp\tilde{\phi}^{(1)})$, are collected in the form

$$\left(1-\frac{V_0}{u}\right)\nabla_\perp\cdot(\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)})+\frac{\nu_{en}c}{B_0\omega_{ce}}\nabla_\perp\cdot(\tilde{n}^{(1)}\nabla_\perp\tilde{\phi}^{(1)}). \tag{23}$$

After using phase velocity relation and Eq. (16), we can see that these two terms cancel each other,

$$-\psi\nabla\cdot(\tilde{n}^{(1)}\nabla\tilde{\chi}^{(1)})+\frac{\nu_{en}\nu_{in}m_ic}{eB_0\omega_{ce}}\nabla_\perp\cdot(\tilde{n}^{(1)}\nabla_\perp\tilde{\chi}^{(1)})=0, \tag{24}$$

where $\nu_{en}\nu_{in}m_ic/eB_0\omega_{ce}=\psi$.

Let us introduce a new variable $\tilde{\phi}^{(1)}=-u\partial F/\partial\xi$. Now, using Eqs. (21) and (22), the remaining equations give a single nonlinear equation for the evolution of FB mode in single-species plasma in the following form:

$$\begin{aligned}&\frac{\partial}{\partial\tau}\nabla^2F-\frac{u}{1+\psi}\frac{c}{B_0}\mathbf{b}\cdot\nabla\frac{\partial F}{\partial\xi}\times\nabla\nabla^2F\\ &+\frac{\psi}{1+\psi}\frac{u^2}{\nu_i}\nabla^2\frac{\partial^2F}{\partial\xi^2}-\frac{\psi}{1+\psi}\frac{(T_e+T_i)}{m_i\nu_i}\nabla^4F\\ &-\frac{e}{2}\frac{\psi}{1+\psi}\frac{u^2}{m_i\nu_i^2}\nabla^2\left(\nabla\frac{\partial}{\partial\xi}F\right)^2=0.\end{aligned} \tag{25}$$

The third term in this equation is destabilizing due to negative diffusion, the fourth term is a standard (positive) diffusion that defines the instability threshold. The second term is a vector nonlinearity obtained in this form as in [18]. The last term is a new nonlinear term of the nonlinear diffusion type.

The cancelation of the scalar convective nonlinear terms in Eq. (25) is a result of a delicate balance of the dissipation that define the phase velocity of the linear waves. Such cancelation does not occur in a plasma with several species which have different collision frequencies. For a plasma consisting of two type of ions, a and b, of different masses and different collisionalities, the governing equations (5), (6), and (13) can be written as

$$\frac{\partial n_{i\alpha}}{\partial t}-\nabla\cdot(n_{i\alpha}\nabla\chi_\alpha)=0, \tag{26}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\nu_\alpha\right)\nabla^2\chi_\alpha\\ &=\nabla^2\left(\frac{e}{m_\alpha}\phi+\frac{T_\alpha}{m_\alpha}\ln\frac{n_{i\alpha}}{n_{0\alpha}}+\frac{(\nabla\chi_\alpha)^2}{2}\right),\end{aligned} \tag{27}$$

where $\alpha=(a, b)$. The lowest order equations (15) and (16) now are

$$-u\partial_\xi\tilde{n}_\alpha^{(1)}-n_{0\alpha}\nabla^2\tilde{\chi}_\alpha^{(1)}=0, \tag{28}$$

$$\nu_\alpha\tilde{\chi}_\alpha^{(1)}=\frac{e}{m_\alpha}\tilde{\phi}^{(1)}. \tag{29}$$

They have to be combined with electron equation (17) by using the quasineutrality condition, $\tilde{n}_e^{(1)}=\tilde{n}_a^{(1)}+\tilde{n}_b^{(1)}$. These equations form the homogeneous system for six variables. The solvability condition for this system defines the phase velocity relation of the linear wave as

$$-1+\frac{V_0}{u}=\psi_{\text{eff}}, \tag{30}$$

where

$$\psi_{\text{eff}}=\frac{\nu_{en}}{\omega_{ce}}\left(\frac{\nu_{en}}{\omega_{ce}}\right)_{\text{eff}}, \tag{31}$$

$$\left(\frac{\nu_{en}}{\omega_{ce}}\right)_{\text{eff}}=\frac{1}{\eta_a\omega_{ca}/\nu_a+\eta_b\omega_{cb}/\nu_b}, \tag{32}$$

with $\eta_\alpha=n_{0\alpha}/n_0$ and $\omega_{c\alpha}=eB_0/m_\alpha c$ for $\alpha=(a, b)$. Note that $\eta_a+\eta_b=1$.

In the second order, with nonlinear and diffusive terms, the equations are

$$\frac{\partial}{\partial\tau}\tilde{n}_\alpha^{(1)}-u\partial_\xi\tilde{n}_\alpha^{(2)}-n_{0\alpha}\nabla^2\tilde{\chi}_\alpha^{(2)}-\nabla\cdot(\tilde{n}_\alpha^{(1)}\nabla\tilde{\chi}_\alpha^{(1)})=0, \tag{33}$$

$$\nu_\alpha\tilde{\chi}_\alpha^{(2)}-u\partial_\xi\tilde{\chi}_\alpha^{(1)}=\frac{e}{m_\alpha}\tilde{\phi}^{(2)}+\frac{T_\alpha}{m_\alpha}\frac{\tilde{n}_\alpha^{(1)}}{n_{0\alpha}}+\frac{1}{2}(\nabla\tilde{\chi}_\alpha^{(1)})^2, \tag{34}$$

$$\frac{\partial}{\partial\tau}\tilde{n}_e^{(1)} + (-u+V_0)\partial_\xi\tilde{n}_e^{(2)} + \frac{\nu_{en}cn_0}{B_0\omega_{ce}}\nabla_\perp^2\tilde{\phi}^{(2)} + \frac{\nu_{en}c}{B_0\omega_{ce}}\nabla_\perp\cdot(\tilde{n}_e^{(1)}\nabla_\perp\tilde{\phi}^{(1)}) - \frac{\nu_{en}cT_e}{eB_0\omega_{ce}}\nabla_\perp^2\tilde{n}_e^{(1)} + \frac{c}{B_0}\mathbf{b}\cdot\nabla\tilde{\phi}^{(1)}\times\nabla\tilde{n}_e^{(1)} = 0. \quad (35)$$

Introducing a new variable $\tilde{\phi}^{(1)} = -u\partial F/\partial\xi$ and using Eqs. (33)–(35) along with dispersion relation (30), the secular terms can be removed and resulting single nonlinear equation for the evolution of FB mode is obtained as

$$\frac{\partial}{\partial\tau}\nabla^2 F - \frac{V_0}{(1+\psi_{\rm eff})^2}\frac{c}{B_0}\mathbf{b}\cdot\nabla\frac{\partial F}{\partial\xi}\times\nabla\nabla^2 F + C_{\rm NL}\nabla\cdot\left(\nabla^2 F\nabla\frac{\partial F}{\partial\xi}\right) + D_I\nabla^2\frac{\partial^2 F}{\partial\xi^2} - D_D\nabla^4 F - C_D\nabla^2\left(\nabla\frac{\partial}{\partial\xi}F\right)^2 = 0 \quad (36)$$

with the coefficients defined as

$$C_{\rm NL} = \frac{V_0}{(1+\psi_{\rm eff})^2}\frac{\nu_{en}c}{B_0\omega_{ce}} \times\left[\frac{\eta_a/m_a^2\nu_a^2 + \eta_b/m_b^2\nu_b^2}{(\eta_a/m_a\nu_a + \eta_b/m_b\nu_b)} - 1\right], \quad (37)$$

$$D_I = \frac{V_0^2\psi_{\rm eff}}{(1+\psi_{\rm eff})^3}\frac{\eta_a/m_a\nu_a^2 + \eta_b/m_b\nu_b^2}{\eta_a/m_a\nu_a + \eta_b/m_b\nu_b}, \quad (38)$$

$$D_D = \frac{1}{1+\psi_{\rm eff}}\frac{\nu_{en}c}{\omega_{ce}eB_0} \times\left[\frac{\eta_a T_a/m_a^2\nu_a^2 + \eta_b T_b/m_b^2\nu_b^2}{(\eta_a/m_a\nu_a + \eta_b/m_b\nu_b)^2} + T_e\right], \quad (39)$$

$$C_D = \frac{e}{2}\frac{V_0^2\psi_{\rm eff}}{(1+\psi_{\rm eff})^3}\frac{\eta_a/m_a^2\nu_a^3 + \eta_b/m_b^2\nu_b^3}{\eta_a/m_a\nu_a + \eta_b/m_b\nu_b}. \quad (40)$$

In case of two ion species, a lowest order scalar nonlinearity term arises with coefficient $C_{\rm NL}$ due to the difference in mass and collision frequency of the two species. This term is zero for $m_a = m_b$ and $\nu_a = \nu_b$, thus reducing to the single ion case as obtained in Eq. (25).

Nonlinear equation (36) can be written in a dimensionless form by introducing the following normalized quantities: $\tau' = \omega_{ca}\tau$ and $\xi' = \xi/\rho_{sa}$ with $\rho_{sa} = c_{sa}/\omega_{ca}$, where $c_{sa} = \sqrt{T_e/m_a}$ and $\omega_{ca} = eB_0/m_ac$. For one-dimensional case, assuming $U = \partial^2F/\partial\xi^2$ and introducing a function f such that $U = \beta f$ with $\beta = 1/2C_{\rm NL}$, we can write Eq. (36) in a form given as

$$\frac{\partial f}{\partial\tau'} + f\frac{\partial f}{\partial\xi'} + D'_\Gamma\frac{\partial^2 f}{\partial\xi'^2} - D'_{\rm NL}\frac{\partial^2 f^2}{\partial\xi'^2} = 0, \quad (41)$$

where the dimensionless coefficients are: $D'_\Gamma = D_\Gamma\omega_{ca}/c_{sa}^2$ with $D_\Gamma = D_I - D_D$ and $D'_{\rm NL} = D_{\rm NL}/\rho_{sa}$ with $D_{\rm NL} = C_D/2C_{\rm NL}$. This form has two nonlinear terms: one of the convective type, as in Burgers equation, responsible for wave breaking; and the another one is the nonlinear diffusion term which is generally stabilizing. The linear term, D_Γ, describes the FB instability for $D_\Gamma > 0$ or diffusive damping for $D_\Gamma < 0$. Typical values of these coefficients for typical plasma parameters for the E-region of ionosphere and the solar chromosphere are given in the Appendix.

4. SUMMARY

We have derived a novel nonlinear equation describing the one-dimensional FB waves in multi-species plasmas. We have shown that one dimensional quadratic nonlinearity persist in plasmas with several ion species, thus affecting the evolution of FB modes and leading to the wave steepening and wave breaking. We have also obtained an additional nonlinear term of the nonlinear diffusion type. The magnitude of the nonlinear coefficients depend on the relative concentration, mass and collision frequencies of ion species as shown in Eq. (36). The situations with several ion species exist in the E-region of the Earth's ionosphere and solar chromosphere. In these cases, the wave steepening nonlinear term may result in wave breaking and forming strongly nonlinear singular solutions. It is worth noting that the FB waves in multispecies plasmas have recently been studied in application to the solar physics problems [33–36]. It has been suggested that FB instability plays an important role in the heating of solar chromosphere [33, 34]. The FB theory has been developed to describe the multi-ion-species metal-dominated plasma of the solar chromosphere [35]. It is shown in this paper that in multispecies plasmas convective nonlinear terms appear that may result in wave breaking of one-dimensional FB waves.

The typical values of the numerical coefficients given in the table show that wave breaking will be more pronounced in the solar chromosphere due to larger mass difference of the ion species while in the E-region of Earth's ionosphere, the nonlinear diffusion effects typically are more important. The two-dimensional (vector) nonlinearity present in Eq. (36) is generally larger than the one-dimensional nonlinear terms for a general case of $k_x \simeq k_y$. However, the linear instability exists in a narrow cone around the direction of the electron $\mathbf{E}\times\mathbf{B}$ flow (along the x axis) and the wave may grow as a one-dimensional structure to a large amplitude before the two-dimensional nonlinear effects will generate waves with finite k_y. Such effects were recently considered in [20]. Therefore, the finite result will depend on the relative competition of two effects: linear growth of one dimensional structures and nonlinear wave-breaking/nonlinear diffusion and

Coefficient in Eq. (41) for the ionosphere and chromosphere plasma parameters

Parameters	Ionosphere E-region		Solar chromosphere	
	105 km	110 km	30 G	105 G
D'_{Γ}	3.61×10^{-3}	1.47×10^{-3}	1.51×10^{-2}	0.10×10^{-2}
$D'_{\rm NL}$	6.02	15.14	0.241	0.502

nonlinear spreading [20], generating finite k_y modes. Numerical analysis of the full equation describing both effects will be presented elsewhere.

APPENDIX

TYPICAL PLASMA PARAMETERS FOR FB INSTABILITY IN THE IONOSPHERE AND SOLAR CHROMOSPHERE

We have used the typical ionosphere parameters at 105 and 110 km altitudes in the E-region of the ionosphere to evaluate the coefficients in Eq. (36). The ion species *a* and *b* are taken to be the two dominant ion species in the E-region of the ionosphere, NO^+ and O_2^+, respectively. Thus, the masses for the two ion species are $m_{NO^+} = 30m_p$ and $m_{O_2^+} = 32m_p$ with $m_p = 1.627 \times 10^{-27}$ kg as the mass of a proton. The electric field and the magnetic field values considered are $E_0 = 0.025$ V/m and $B_0 = 0.5$ G, which gives the drift velocity $V_0 = 500$ m/s. The cyclotron frequency for electrons is $\omega_{ce} = 8.79 \times 10^6$ s^{-1} and the cyclotron frequencies for each of the ions species are $\omega_{ca} = 1.64 \times 10^2$ s^{-1} (for NO^+) and $\omega_{cb} = 1.54 \times 10^2$ s^{-1} (for O_2^+). We consider equal temperatures both for the electrons and the ion species, so $T_e = T_a = T_b = 300$ K, which gives $c_{sa} = 291.24$ m/s. To find the collision frequencies of each of the charged species with the neutrals, we consider the nonresonant ion–neutral collision frequencies only. We took the simple expression for nonresonant collision frequency for a given ion–neutral pair as given in [37], which is $\nu_{in} = C_{in}n_n$ with C_{in} as a numerical collision frequency coefficient and n_n as the density of neutrals in cm^{-3}. The neutral species we have considered are N_2 and O_2 molecules. The density of the neutrals is taken from the MSIS atmospheric model [http://ccmc.gsfc.nasa.gov/modelweb/models/msis_vitmo.php] (for March 17, 2014 with universal time (hour = 1.5) and geographic coordinates at the latitude 55° and longitude 45°). It should be noted that we have not taken into account the resonant collision frequency of O_2^+ ions with O_2 neutrals as the resonant ion–neutral collision frequency values become important only at higher altitudes and for temperature values greater than 300 K.

For the 105-km altitude, the densities obtained for N_2 and O_2 are 3.645×10^{12} cm^{-3} and 7.552×10^{11} cm^{-3}, respectively. The collision frequency coefficients are taken from [37] as C_{in} (NO^+–N_2) = 4.34×10^{-10} cm^3 s^{-1}, C_{in} (NO^+–O_2) = 4.27×10^{-10} cm^3 s^{-1}, and C_{in} (O_2^+–N_2) = 4.13×10^{-10} cm^3 s^{-1}. So, the ion–neutral collision frequencies obtained for the two ion species are $\nu_{NO^+-N_2} = 1.58 \times 10^3$ s^{-1} and $\nu_{NO^+-O_2} = 3.22 \times 10^2$ s^{-1}, which gives the total collision frequency of NO^+ ions with neutrals to be 1.90×10^3 s^{-1} and $\nu_{O_2^+-N_2} = 1.51 \times 10^3$ s^{-1}. The collision frequency of electrons with neutrals along the E-region of ionosphere follows $\nu_{en} = 10\nu_{in}$. So, using the total ion–neutral frequency calculated above for both ion species, the electron-neutral collision frequency is $\nu_{en} = 3.41 \times 10^4$ s^{-1}. To find the η values for both the species, we need the densities for both the ions and the electrons. So, using the modeled ion densities and the electron density profile as in [38, 39] for NO^+ and O_2^+ at 105 km, we obtain $\eta_a = \eta_{NO^+} = \eta_{NO^+}/\eta_{e0} = 0.5$ and $\eta_b = \eta_{O_2^+} = \eta_{O_2^+}/n_{e0} = 0.5$. Similarly, for the two ion species at the 110-km altitude, the values obtained are $\eta_a = 0.6$, $\eta_b = 0.4$; $\nu_{NO^+-N_2} = 6.05 \times 10^2$ s^{-1}, and $\nu_{NO^+-O_2} = 1.05 \times 10^2$ s^{-1}, which gives the total collision frequency of NO^+ ions with neutrals to be 7.10×10^2 s^{-1}; $\nu_{O_2^+-N_2} = 5.76 \times 10^2$ s^{-1}, and, thus, $\nu_{en} = 1.29 \times 10^4$ s^{-1}. Using all these parameters, the normalized coefficient values for Eq. (41) are given in the table.

In the solar chromosphere, the FB instability may develop near the solar temperature minimum, where electrons are strongly magnetized but the protons and the heavy ions are unmagnetized [33–36]. For the chromosphere parameters, just below the temperature minimum, MgII, SiII, and FeII are the dominant ion species, while protons and CII (to a lesser extent) dominate above the temperature minimum. We have considered electrons and the two ion species: (*a*) protons and (*b*) FeII. Thus, for the mass of the ions, we have: $m_a = 1.627 \times 10^{-27}$ kg for protons and $m_b = 9.09 \times 10^{-26}$ kg for FeII. The coefficients are calculated using the various parameters taken from the plots of collision frequency, plasma density, temperature profile, and trigger velocities in [35] corresponding to magnetic field values of 30 and 105 G near the solar temperature minimum.

We first consider parameters at temperature minimum for the magnetic field $B_0 = 30$ G. The trigger velocity and the solar temperature minimum values corresponding to this magnetic field are 8 km/s and 3800 K, respectively. We will take V_0 to be greater than the trigger velocity, $V_0 = 14$ km/s, and the temperatures $T_e = T_a = T_b$ equal to the temperature minimum, which gives $c_{sa} = 5.68$ km/s. The ratios of the collision frequency to the gyrofrequency for electrons and ions are $\nu_{en}/\omega_{ce} = 0.07$, $\nu_{an}/\omega_{ca} = 9$, and $\nu_{bn}/\omega_{cb} = 100$. The density values are $n_a = 4 \times 10^{16}$ m^{-3}, $n_b = 0.7 \times 10^{16}$ m^{-3}, and $n_e = 5 \times 10^{16}$ m^{-3}, which satisfies the quasineutrality assumption $n_e \approx n_a + n_b$. Using these density values, we have $\eta_a = 0.8$ and $\eta_b = 0.2$. Similarly, we consider parameters at same temperature minimum for the magnetic field $B_0 = 105$ G. The trigger velocity corresponding to this magnetic field value now is 5.2 km/s, so we take $V_0 = 8$ km/s. The ratios of the collision frequency to the gyrofrequency are $\nu_{en}/\omega_{ce} = 0.02$, $\nu_{an}/\omega_{ca} = 4$, and $\nu_{bn}/\omega_{cb} = 40$. The values for c_{sa}, η_a, and η_b remain the same. Using these values, the calculated normalized coefficients in Eq. (41) are given in table.

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