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Drift kinetic equation in the moving reference frame and reduced magnetohydrodynamic equations

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The drift kinetic equation is formulated by using the phase space conservation law and drift equations of particle motion in the reference frame moving with plasma fluid velocity. The latter includes the parallel and ExB drift, diamagnetic velocity, and diamagnetic velocity due to the parallel viscosity tensor (anisotropic pressure). It is shown explicitly that the particle drift equations conserve the adiabatic invariant and kinetic equation conserves the phase space volume. The resulting drift kinetic equation is used to obtain a set of moment equations corresponding to the conservation laws for plasma density, momentum, and energy. These equations are compared with reduced equations obtained from hierarchy of extended magnetohydrodynamic equations including the evolution equation for the heat flux (Grad hydrodynamics). The reduction is done in the drift limit by using $1/B$ as an expansion parameter. We demonstrate that reduced moment equations derived from our drift kinetic equation are identical to the reduced equations obtained from extended magnetohydrodynamic equations. The structure of the reduced equations and implications for the closure problem, including neoclassical effects, are discussed. © 2010 American Institute of Physics. [doi:10.1063/1.3360297]

I. INTRODUCTION

The fluid equations often provide a simple but insightful approach for analytical and numerical studies of plasmas. The reduced fluid equations for low frequency $\omega \ll \omega_{ci}$, long wavelength perturbations $k_{\perp}^2 \rho_i^2 \ll 1$, where $\rho_i^2 = v_{Ti}^2 / \omega_{ci}^2$, $v_{Ti}^2 = 2T_i/m_i$, can be obtained from the moments of the drift kinetic equation or directly from full hierarchy of moment equations obtained from Vlasov (Boltzmann) equation. Both procedures are based on the asymptotic reduction of an initial system by using strong magnetic field as an expansion parameter, which translates into low frequency $\omega \ll \omega_{ci}$ long wavelength perturbations $k_{\perp}^2 \rho_i^2 \ll 1$ approximations. Reduced system of fluid moments, e.g., for density, velocity, and pressure, in general is not able to describe kinetic effects such as wave-particle (Landau) interactions or neoclassical transport. However, it is possible to include some of these effects via nonasymptotic closure terms. Neoclassical transport has been incorporated into the moment approach via kinetic calculations of higher order moments such as parallel viscosity and heat flux energy weighted viscosity tensors.¹⁻⁴ Linear Landau damping effects were also included via the heat flux and parallel viscosity terms.⁵⁻⁹ Toroidal effects have also been added to Landau closures,¹⁰⁻¹³ but it is not clear whether these closures reproduce standard neoclassical transport. Closures that would describe wave-particle resonances, finite ion Larmor radius, and neoclassical effects within a uniform approach have been of considerable interest now,¹⁴⁻¹⁹ in particular, for numerical simulations with extended magnetohydrodynamics (MHD) models aiming to include kinetic and toroidal effects.²⁰⁻²³

In all the above applications, it is important to ensure

that the initial set of moment equations and the kinetic equation used to calculate the closures for higher order moments are consistent with each other, e.g., that the basic conservation laws (of density, momentum, and energy) are respected in the relevant drift kinetic equation to the appropriate order. Moreover, often, it is not even obvious which higher order moments will have to be chosen for closures, e.g., the Landau damping closures can be formulated in several different forms.⁵⁻⁷

There are several important papers devoted to the derivation of the drift kinetic equation. We are using the transformation to the moving reference frame and our resulting drift kinetic equation in a number of ways is similar to those derived in Refs. 24-26. Contrary to Refs. 24-26, which use the direct (but cumbersome) asymptotic expansion procedure for Vlasov equation, our drift kinetic equation is simply formulated on a basis of the averaged equations for particle motion. It is worth noting that the averaging (over fast cyclotron rotation) is done after the transformation to a moving reference frame. In fact, our resulting drift kinetic equation is identical to a low pressure limit of the drift kinetic equation of Ref. 26.

There are several objectives of this paper. First, we present a novel formulation of the drift kinetic equation and derive the reduced fluid equations by taking the moments of such an equation. Next, we perform the expansion of the hierarchy of the moments of the full (nonaveraged) Vlasov equation (in particular, the Grad equation for the evolution of the heat flux) and show that the resulting equations are identical to the reduced fluid equations obtained from our drift kinetic equation. Finally, we discuss the structure of the remaining higher order moments for which the closures have to be calculated from the from the initial drift kinetic equation.

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The paper is organized as follows. In Sec. II we derive the drift kinetic equation and discuss its properties. In Sec. III, we obtain reduced fluid equations as moments of the drift kinetic equation. In Sec. IV, we derive the reduced fluid equations by asymptotic expansion of the hierarchy of Vlasov equation. The results are summarized and discussed in Sec. V.

II. DRIFT KINETIC EQUATION IN A MOVING REFERENCE FRAME

In this work, we do not perform straightforward (but quite tedious) procedure of asymptotic averaging of the initial Vlasov equation as is customary done in derivations of the drift kinetic equation.^{24–26} In our approach, we average equations of particle motion in the moving reference frame and use these equations to formulate the drift kinetic equation in the phase space conserving form.

A. Equations of particle motion in a moving reference frame

We start from the initial equation for particles motion,

$$\frac{d}{dt}\mathbf{v} = \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right). \quad (1)$$

We introduce a particle velocity $\mathbf{w} = \mathbf{v} - \mathbf{V}$, which is a “random” particle velocity in the reference frame moving with the velocity $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$, \mathbf{v} is the total particle velocity. Then the equation of particle motion takes the form

$$\frac{d}{dt}\mathbf{w} = \frac{e}{m}\mathbf{w} \times \mathbf{B} - \mathbf{w} \cdot \nabla \mathbf{V} + \mathbf{F}, \quad (2)$$

where the function $\mathbf{F}(\mathbf{r}, t)$ is defined by the expression

$$\mathbf{F}(\mathbf{r}, t) \equiv \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B}\right) - \frac{D\mathbf{V}}{Dt}. \quad (3)$$

The fluid derivative in Eq. (3) is formally introduced here as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla. \quad (4)$$

Equation (2) represents the particle motion in a reference frame moving with the velocity $\mathbf{V}(\mathbf{r}, t)$, where the function $\mathbf{F}(\mathbf{r}, t)$ is a generalized inertial force. Equation (2) has a large parameter $\omega_c/(d/dt)$ ($\omega_c = qB/mc$ is the particle cyclotron frequency), which allows asymptotic averaging. We use a procedure in the form formulated by Morozov.²⁷ In the low β limit ($\nabla \times \mathbf{B} \approx 0$), the averaged (drift) equations of motion are

$$\frac{d}{dt}\mathbf{r}' = w_{\parallel}\mathbf{b} + \frac{1}{\omega_c}\mathbf{F} \times \mathbf{b} + \frac{1}{\omega_c}\left(\frac{w_{\perp}^2}{2} + w_{\parallel}^2\right)\mathbf{b} \times \nabla \ln B, \quad (5)$$

$$\frac{d}{dt}w_{\parallel} = \mathbf{F} \cdot \mathbf{b} + \frac{w_{\perp}^2}{2}\nabla \cdot \mathbf{b} + \frac{w_{\parallel}}{\omega_c}\mathbf{F} \cdot \mathbf{b} \times \nabla \ln B - w_{\parallel}\mathbf{b}\mathbf{b}:\nabla \mathbf{V}, \quad (6)$$

$$\begin{aligned} \frac{d}{dt}\frac{w_{\perp}^2}{2} = & -\frac{w_{\perp}^2 w_{\parallel}}{2}\nabla \cdot \mathbf{b} - \frac{w_{\perp}^2}{2\omega_c}\mathbf{b} \cdot \nabla \times \mathbf{F} + \frac{w_{\perp}^2}{2\omega_c}\mathbf{F} \cdot \mathbf{b} \\ & \times \nabla \ln B - \frac{w_{\perp}^2}{2}(\nabla \cdot \mathbf{V} - \mathbf{b}\mathbf{b}:\nabla \mathbf{V}). \end{aligned} \quad (7)$$

Here $d\mathbf{r}'/dt$ is the drift velocity of the guiding center in the moving frame and \mathbf{b} is the unit vector along the magnetic field. Terms of the order $1/\omega_c^2$ and higher (including inertial drift) have been neglected here.

Equations (6) and (7) can be rewritten in terms of the kinetic energy, $\mathcal{E} = w^2/2$, and the magnetic moment, $\mu = w_{\perp}^2/2B$,

$$\begin{aligned} \frac{d}{dt}\frac{mw^2}{2} = & mw_{\parallel}\mathbf{F} \cdot \mathbf{b} + m\frac{w_{\perp}^2/2 + w_{\parallel}^2}{\omega_c}\mathbf{F} \cdot \mathbf{b} \times \nabla \ln B \\ & - \frac{mw_{\perp}^2}{2\omega_c}\mathbf{b} \cdot \nabla \times \mathbf{F} - \frac{mw_{\perp}^2}{2}\nabla \cdot \mathbf{V} \\ & - \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2}\right)\mathbf{b}\mathbf{b}:\nabla \mathbf{V}, \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{d}{dt}\frac{w_{\perp}^2}{2B} = & -\frac{w_{\perp}^2}{2B\omega_c}\mathbf{b} \cdot \nabla \times \mathbf{F} - \frac{w_{\perp}^2}{2B}\mathbf{V} \cdot \nabla \ln B \\ & - \frac{w_{\perp}^2}{2B}(\nabla \cdot \mathbf{V} - \mathbf{b}\mathbf{b}:\nabla \mathbf{V}). \end{aligned} \quad (9)$$

These are complete equations of particle motion averaged to the order of ω_c^{-2} terms in the low plasma pressure approximation.

B. Drift kinetic equation

In the above representation, the total particle velocity is

$$\mathbf{v} = \mathbf{V}(\mathbf{r}, t) + \mathbf{w}. \quad (10)$$

We impose a condition that $\mathbf{V}(\mathbf{r}, t)$ be the standard fluid velocity as determined by fluid moment equations. Then the random velocity \mathbf{w} is determined as a difference between the total velocity and the fluid velocity (of the moving reference frame). In other words, we postulate that the ensemble averaging is defined as a reduction of \mathbf{v} to $\mathbf{V}(\mathbf{r}, t)$.

The required distribution function is of the form $f = f(w_{\parallel}, w_{\perp}^2, \mathbf{r}, t)$, and in the moving reference frame satisfies the drift kinetic equation in the phase space conserving form²⁸

$$\begin{aligned} \frac{\partial f}{\partial t} + \nabla \cdot \left[\left(\frac{d\mathbf{r}'}{dt} + \mathbf{V} \right) f \right] + \frac{\partial}{\partial w_{\parallel}} \left(\frac{dw_{\parallel}}{dt} f \right) + \frac{\partial}{\partial w_{\perp}^2} \left(\frac{dw_{\perp}^2}{dt} f \right) \\ = 0. \end{aligned} \quad (11)$$

It can be readily checked that averaged equations of particle motion (5)–(7) conserve the phase space volume in the form

$$\nabla \cdot \left(\frac{d\mathbf{r}'}{dt} + \mathbf{V} \right) + \frac{\partial}{\partial v_{\parallel}} \left(\frac{dv_{\parallel}}{dt} \right) + \frac{\partial}{\partial v_{\perp}^2} \left(\frac{dv_{\perp}^2}{dt} \right) = 0. \quad (12)$$

The phase space volume is conserved for arbitrary \mathbf{F} , i.e., for the arbitrary velocity $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$. It is worth noting that Eq.

(11) is uniquely determined by the drift equations (5)–(7) since they are characteristics of Eq. (11).

Alternately, the drift kinetic equation can be presented in μ and \mathcal{E} variables

$$\frac{\partial f}{\partial t} + \left(\frac{d\mathbf{r}'}{dt} + \mathbf{V} \right) \cdot \nabla f + \frac{d\mu}{dt} \frac{\partial f}{\partial \mu} + \frac{d\mathcal{E}}{dt} \frac{\partial f}{\partial \mathcal{E}} = 0, \quad (13)$$

where the evolution of μ and \mathcal{E} is defined by Eqs. (8) and (9).

C. Conservation of the adiabatic invariant

So far the velocity of the moving frame $\mathbf{V}=\mathbf{V}(\mathbf{r},t)$ in Eqs. (5)–(7) was an arbitrary function in time and space. The definition of the adiabatic moment depends on this velocity so we have to introduce it now. We introduce the velocity $\mathbf{V}=\mathbf{V}(\mathbf{r},t)$ as the total fluid velocity of the respective plasma component that satisfies the momentum balance equation

$$m n \frac{D\mathbf{V}}{Dt} = e n \left(\mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{B} \right) - \nabla p - \nabla \cdot \mathbf{\Pi}. \quad (14)$$

Together with Eq. (3), it gives for the function \mathbf{F}

$$\mathbf{F} = \frac{1}{m n} (\nabla p + \nabla \cdot \mathbf{\Pi}). \quad (15)$$

From the momentum balance equation (14), we determine the total plasma flow to the first order in $1/B$ in the form

$$\mathbf{V} = V_{\parallel} \mathbf{b} + \mathbf{V}_E + \mathbf{V}_p + \mathbf{V}_{\pi}, \quad (16)$$

where the $\mathbf{E} \times \mathbf{B}$ drift, diamagnetic and viscosity components are

$$\mathbf{V}_E = \frac{c}{B} \mathbf{E} \times \mathbf{b}, \quad \mathbf{V}_p = \frac{c}{e n B} \mathbf{b} \times \nabla p, \quad (17)$$

$$\mathbf{V}_{\pi} = \frac{c}{e n B} \mathbf{b} \times \nabla \cdot \mathbf{\Pi}.$$

The viscosity is represented by the parallel component

$$\mathbf{\Pi} = \mathbf{\Pi}_{\parallel} = \frac{3}{2} \pi_{\parallel} (\mathbf{b}\mathbf{b} - \frac{1}{3} \mathbf{I}). \quad (18)$$

The contribution of the gyroviscosity which is of the second order in $1/B$ is neglected here. Using Eq. (9) and identities (C1)–(C10), we obtain

$$\frac{d w_{\perp}^2}{dt 2B} = 0. \quad (19)$$

It can be readily seen that $\mathbf{b} \cdot \nabla \times \mathbf{F} / \omega_c$ term is important for the conservation of the adiabatic moment when the diamagnetic, \mathbf{V}_p , and viscous, \mathbf{V}_{π} , drifts are taken into account. In the absence of diamagnetic and viscous effects when $\mathbf{V}_{\perp} = \mathbf{V}_E$, this term identically vanishes.

III. MOMENTS OF THE DRIFT KINETIC EQUATION

A. Momentum and density conservation

The momentum balance equation is satisfied automatically because it is used as a definition for the fluid velocity \mathbf{V} . Momentum conservation has the form (14), where $\mathbf{\Pi}$ includes only the parallel viscosity. It is easy to see that once

the velocity of the reference frame has been chosen to be the fluid velocity $\mathbf{V}(\mathbf{r},t)$, the continuity equation automatically takes the standard form

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}) = 0. \quad (20)$$

It is equivalent to the condition

$$\nabla \cdot \left\{ \int f(\mathbf{r},t,w) d\mathbf{w} \left[\frac{1}{\omega_c} \mathbf{F} \times \mathbf{b} + \frac{1}{\omega_c} \left(\frac{w_{\perp}^2}{2} + w_{\parallel}^2 \right) \times \mathbf{b} \times \nabla \ln B \right] \right\} = 0. \quad (21)$$

It is easy to see that this is satisfied when the expression (15) is used for \mathbf{F} and standard definitions for plasma pressure and parallel viscosity are employed,

$$\frac{3}{2} p = \left\langle \frac{m w^2}{2} \right\rangle, \quad (22)$$

$$\frac{3}{2} \pi_{\parallel} = (p_{\parallel} - p_{\perp}) = m \left\langle w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right\rangle, \quad (23)$$

where the angle brackets mean

$$\langle \dots \rangle = \int (\dots) f(\mathbf{r},t,w) d\mathbf{w}.$$

By using expression (16) in Eq. (20), the continuity equation is reduced to the form²⁹

$$\frac{\partial n}{\partial t} + \nabla \cdot \left[n V_{\parallel} \mathbf{b} + n \mathbf{V}_E + \frac{1}{m \omega_c} \mathbf{b} \times \nabla \ln B \left(2p + \frac{1}{2} \pi_{\parallel} \right) \right] = 0. \quad (24)$$

B. Energy balance equation

Energy balance equation is obtained by averaging drift kinetic equation with a weight $m w^2 / 2$ giving

$$\frac{\partial}{\partial t} \left\langle \frac{m w^2}{2} \right\rangle + \nabla \cdot \left\langle \frac{m w^2}{2} \left(\mathbf{V} + \frac{d\mathbf{r}'}{dt} \right) \right\rangle - \left\langle \frac{d m w^2}{dt 2} \right\rangle = 0. \quad (25)$$

From Eq. (8) and using \mathbf{V} in the form (16), one obtains

$$\begin{aligned} \left\langle \frac{d m w^2}{dt 2} \right\rangle &= \left(2p + \frac{1}{2} \pi_{\parallel} \right) \mathbf{V}_E \cdot \nabla \ln B \\ &\quad - \left(p - \frac{1}{2} \pi_{\parallel} \right) V_{\parallel} \nabla \cdot \mathbf{b} - (p + \pi_{\parallel}) \mathbf{b} \cdot \nabla V_{\parallel}. \end{aligned} \quad (26)$$

The average of the random particle velocity with weight $m w^2 / 2$ is

$$\left\langle \frac{mw^2 d\mathbf{r}'}{2 dt} \right\rangle = q_{\parallel} \mathbf{b} + \frac{3p}{2\omega_c} \mathbf{F} \times \mathbf{b} + \frac{1}{\omega_c} \left(5\frac{pT}{m} + \frac{7T}{4m} \pi_{\parallel} + \frac{1T}{2m} \pi_{\parallel}^* + \frac{2T}{3m} \chi \right) \mathbf{b} \times \nabla \ln B, \quad (27)$$

where the respective moments are defined as follows:

$$q_{\parallel} = \frac{m}{2} \int w_{\parallel} w^2 f d^3 w, \quad (28)$$

$$\chi = mv_T^2 \int \left(\frac{w^4}{v_T^4} - 5\frac{w^2}{v_T^2} + \frac{15}{4} \right) f d^3 v, \quad (29)$$

$$\frac{3}{2} \pi_{\parallel}^* = m \int \left(\frac{w^2}{v_T^2} - \frac{7}{2} \right) \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) f d^3 w. \quad (30)$$

We have used here the following representation in terms of the irreducible higher order moments:

$$\begin{aligned} w^2 \left(\frac{w_{\perp}^2}{2} + w_{\parallel}^2 \right) &= \frac{10}{3} v_T^4 \left(\frac{w^2}{v_T^2} - \frac{3}{4} \right) + \frac{7}{6} v_T^2 \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \\ &+ \frac{1}{3} v_T^2 \left(\frac{w^2}{v_T^2} - \frac{7}{2} \right) \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \\ &+ \frac{2}{3} v_T^4 \left(\frac{w^4}{v_T^4} - 5\frac{w^2}{v_T^2} + \frac{15}{4} \right). \end{aligned} \quad (31)$$

The final energy balance equation takes the form

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{3}{2} p \right) + \nabla \cdot \left(\frac{3}{2} p V_{\parallel} \mathbf{b} + \frac{3}{2} p \mathbf{V}_E + q_{\parallel} \mathbf{b} \right. \\ \left. + \frac{1}{\omega_c} \left(5\frac{pT}{m} + \frac{7T}{4m} \pi_{\parallel} + \frac{1T}{2m} \pi_{\parallel}^* + \frac{2T}{3m} \chi \right) \mathbf{b} \times \nabla \ln B \right) \\ - \left(2p + \frac{1}{2} \pi_{\parallel} \right) \mathbf{V}_E \cdot \nabla \ln B \\ + \left(p - \frac{1}{3} (p_{\parallel} - p_{\perp}) \right) V_{\parallel} \nabla \cdot \mathbf{b} \\ + \left(p + \frac{2}{3} (p_{\parallel} - p_{\perp}) \right) \mathbf{b} \cdot \nabla V_{\parallel} = 0. \end{aligned} \quad (32)$$

Note here that three higher order moments, π_{\parallel} , π_{\parallel}^* , and χ , corresponding to the irreducible representation of the energy flux due to magnetic drift velocity in (31), are responsible for the neoclassical energy flux.

C. Viscosity evolution equation

An equation for the evolution of parallel viscosity is obtained by averaging the drift kinetic equation with a weight $(w_{\parallel}^2 - w_{\perp}^2/2)$ giving

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle m \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \right\rangle + \nabla \cdot \left\langle m \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \left(\mathbf{V} + \frac{d\mathbf{r}'}{dt} \right) \right\rangle \\ - \left\langle m \frac{d}{dt} \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \right\rangle = 0. \end{aligned} \quad (33)$$

By using Eqs. (6) and (7), one obtains

$$\begin{aligned} \left\langle \frac{d}{dt} m \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \right\rangle \\ = \frac{m \langle w_{\perp}^2 \rangle}{2} \left(\frac{1}{\omega_c} \mathbf{b} \cdot \nabla \times \mathbf{F} - \frac{1}{\omega_c} \mathbf{F} \cdot \mathbf{b} \times \nabla \ln B \right. \\ \left. + \nabla \cdot \mathbf{V} - \mathbf{b} \mathbf{b} : \nabla \mathbf{V} \right) \frac{3}{2} m \langle w_{\perp}^2 w_{\parallel} \rangle \nabla \cdot \mathbf{b} + 2m \langle w_{\parallel}^2 \rangle \\ \times \left(\frac{1}{\omega_c} \mathbf{F} \cdot \mathbf{b} \times \nabla \ln B - \mathbf{b} \mathbf{b} : \nabla \mathbf{V} \right). \end{aligned} \quad (34)$$

Using the definitions (15) and (16) and identities (C1)–(C8) and (34) reduces to

$$\begin{aligned} \left\langle \frac{d}{dt} m \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \right\rangle \\ = \left(p + \frac{5}{2} \pi_{\parallel} \right) \mathbf{V}_E \cdot \nabla \ln B + \left(p - \frac{1}{2} \pi_{\parallel} \right) V_{\parallel} \nabla \cdot \mathbf{b} \\ - 2(p + \pi_{\parallel}) \mathbf{b} \cdot \nabla V_{\parallel} + 3q_{\parallel}^{\perp} \nabla \cdot \mathbf{b}. \end{aligned} \quad (35)$$

The average of the random particle velocity with the weight $m(w_{\parallel}^2 - w_{\perp}^2/2)$ is reduced to the form

$$\begin{aligned} \left\langle m \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \frac{d\mathbf{r}'}{dt} \right\rangle \\ = (q_{\parallel}^{\parallel} - q_{\parallel}^{\perp}) \mathbf{b} + \frac{3\pi_{\parallel}}{2\omega_c} \mathbf{F} \times \mathbf{b} + \frac{1}{\omega_c} \\ \times \left(\frac{pT}{m} + 8\frac{T}{m} \pi_{\parallel} + \frac{16T}{7m} \pi_{\parallel}^* + \frac{2T}{15m} \chi + \frac{3T}{2m} \lambda \right) \\ \times \mathbf{b} \times \nabla \ln B. \end{aligned} \quad (36)$$

Here, the higher order moments are introduced by definitions

$$q_{\parallel}^{\parallel} = m \int w_{\parallel} w_{\parallel}^2 f d^3 w, \quad (37)$$

$$q_{\parallel}^{\perp} = \frac{m}{2} \int w_{\parallel} w_{\perp}^2 f d^3 w, \quad (38)$$

$$\lambda = \frac{1}{2T} \int \left(w_{\parallel}^4 - \frac{6}{7} w_{\parallel}^2 w^2 + \frac{3}{35} w^4 \right) f d^3 v. \quad (39)$$

It is worth noting that the higher order moments λ , χ , and π_{\parallel}^* are required to describe the general case of the fourth rank anisotropy. The following identity has been used to represent the $(w_{\parallel}^4 - w_{\perp}^4/4)$ combination in the form of the irreducible moments

$$\begin{aligned} \left(w_{\parallel}^4 - \frac{w_{\perp}^4}{4} \right) &= \frac{3}{4} \left(w_{\parallel}^4 - \frac{6}{7} w_{\parallel}^2 w^2 + \frac{3}{35} w^4 \right) + \frac{16}{21} v_T^2 \left(\frac{w^2}{v_T^2} - \frac{7}{2} \right) \\ &\times \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) + \frac{8}{3} v_T^2 \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2} \right) \\ &+ \frac{1}{15} v_T^4 \left(\frac{w^4}{v_T^4} - 5\frac{w^2}{v_T^2} + \frac{15}{4} \right) + \frac{1}{3} v_T^4 \left(\frac{w^2}{v_T^2} - \frac{3}{4} \right). \end{aligned} \quad (40)$$

Finally, the evolution equation for the parallel viscosity takes the form

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{3}{2} \pi_{\parallel} \right) + \nabla \cdot \left(\frac{3}{2} \pi_{\parallel} V_{\parallel} \mathbf{b} + \frac{3}{2} \pi_{\parallel} \mathbf{V}_E + (q_{\parallel}^{\parallel} - q_{\parallel}^{\perp}) \mathbf{b} \right. \\ & + \frac{1}{\omega_c} \left(\frac{pT}{m} + 8 \frac{T}{m} \pi_{\parallel} + \frac{16T}{7m} \pi_{\parallel}^* + \frac{2T}{15m} \chi + \frac{3T}{2m} \lambda \right) \\ & \times \mathbf{b} \times \nabla \ln B \left. \right) - \left(p + \frac{5}{2} \pi_{\parallel} \right) \mathbf{V}_E \cdot \nabla \ln B \\ & + \left(p - \frac{1}{2} \pi_{\parallel} \right) V_{\parallel} \nabla \cdot \mathbf{b} + 2(p + \pi_{\parallel}) \mathbf{b} \cdot \nabla V_{\parallel} - 3q_{\parallel}^{\perp} \nabla \cdot \mathbf{b} \\ & = 0. \end{aligned} \quad (41)$$

In the next section, we consider how the moment equations derived from the drift kinetic equation are related to the extended magnetohydrodynamics equations.

IV. REDUCED MAGNETOHYDRODYNAMIC EQUATIONS

In this section, we derive the reduced transport equations from the extended hierarchy of magnetohydrodynamic equations, which includes the evolution equation for the heat flux (Grad equation). The plasma continuity equation is identical to Eq. (20) obtained from the drift kinetic equation, and respectively, the reduced form is given by Eq. (24). The energy balance equation has a standard form given by Braginskii

$$\frac{3}{2} \frac{\partial}{\partial t} p + \nabla \cdot \left(\frac{3}{2} p \mathbf{V} \right) + p \nabla \cdot \mathbf{V} + \Pi : \nabla \mathbf{V} + \nabla \cdot \mathbf{q} = 0. \quad (42)$$

As it has been noted above, parallel components of \mathbf{V} , \mathbf{q} , and Π have to be retained without reduction. The perpendicular components however can be obtained by the expansion of the evolution equations for \mathbf{V} , \mathbf{q} , and Π . In the first order $\mathcal{O}(1/B)$, the perpendicular fluid velocity \mathbf{V} is given by Eq. (16). The gyroviscosity can also be found from the evolution equation for Π ,³⁰ but it will be neglected here as the second order effect in $1/B$. Only, the parallel viscosity will be used for Π in Eq. (42).

The perpendicular component of the heat flux required in Eq. (42) can be determined from the Grad-type equation for the evolution of the heat flux.³⁰ Such equation was derived in Ref. 31, see also Ref. 32. These papers use different definitions, also there are some typos. For completeness and to fix the notations, we give the derivations of the heat flux equation in Appendix A. To the first order in $1/B$, the equation for \mathbf{q} has the form (see Appendix A)

$$\begin{aligned} \omega_c \mathbf{q} \times \mathbf{b} = \nabla \cdot \left(\frac{5pT}{2m} \mathbf{I} + \frac{7}{4} v_T^2 \Pi + \frac{1}{2} v_T^2 \Pi^* + \frac{1}{6} v_T^2 \chi \mathbf{I} \right) \\ - \mathbf{F} \cdot \Pi - \frac{5}{2} \mathbf{F} p, \end{aligned} \quad (43)$$

where the higher order scalar moment χ is given by the expression (29). The higher order viscosity tensor Π^* is defined as

$$\Pi^* = m \int \left(\frac{w^2}{v_T^2} - \frac{7}{2} \right) \left(\mathbf{w} \mathbf{w} - \frac{1}{3} w^2 \mathbf{I} \right) f d^3 w, \quad (44)$$

and

$$\Pi = m \int \left(\mathbf{w} \mathbf{w} - \frac{1}{3} w^2 \mathbf{I} \right) f d^3 w. \quad (45)$$

Again evolution equation can be written for Π and Π^* and perpendicular components of these tensors can be determined by the expansion in $1/B$. However, in the first order the perpendicular components are neglected, and only the parallel components have to be retained. The tensor Π_{\parallel}^* is defined similarly to Π_{\parallel} ,

$$\Pi_{\parallel}^* = \frac{3}{2} \pi_{\parallel}^* \left(\mathbf{b} \mathbf{b} - \frac{1}{3} \mathbf{I} \right). \quad (46)$$

Note that the tensor Θ_{\parallel} , which typically occurs in neoclassical theory,¹ is related to Π_{\parallel}^* via the equation

$$\Theta_{\parallel} = \Pi_{\parallel}^* + \Pi_{\parallel}.$$

From Eq. (43), we find the heat flux to the first order as

$$\mathbf{q} = \mathbf{q}_{\lambda} + \mathbf{q}_{\pi} + \mathbf{q}_{\pi^*} + \mathbf{q}_{\chi}, \quad (47)$$

where \mathbf{q}_{λ} is the standard collisionless heat flux by Braginskii,

$$\mathbf{q}_{\lambda} = \frac{5}{2} \frac{p}{m \omega_c} \mathbf{b} \times \nabla T. \quad (48)$$

The other components of the collisionless heat flux \mathbf{q}_{π} , \mathbf{q}_{π^*} , and \mathbf{q}_{χ} are related to higher order moments

$$\begin{aligned} \mathbf{q}_{\pi} = - \frac{5}{2} \frac{T}{m \omega_c} \mathbf{b} \times \nabla \cdot \Pi + \frac{7}{2 \omega_c} \mathbf{b} \times \nabla \cdot \left(\frac{T}{m} \Pi \right) \\ - \frac{1}{\omega_c} \mathbf{b} \times \mathbf{F} \cdot \Pi, \end{aligned} \quad (49)$$

$$\mathbf{q}_{\pi^*} = \frac{1}{\omega_c} \mathbf{b} \times \nabla \cdot \left(\frac{T}{m} \Pi^* \right), \quad (50)$$

$$\mathbf{q}_{\chi} = \frac{1}{3 \omega_c} \mathbf{b} \times \nabla \cdot \left(\frac{T}{m} \chi \mathbf{I} \right). \quad (51)$$

Using Eqs. (16), (18), and (47)–(51) in Eq. (42) one obtains the exact form of Eq. (32) (further details are given in Appendix B).

V. SUMMARY AND DISCUSSION

We have presented a new approach to the derivation of the drift kinetic equation in the moving reference frame. In this approach, we do not average the initial full Vlasov equation. It has been shown here that drift kinetic equation can be simply formulated based on the conservation of phase space and the characteristic equations, which are defined by the averaged equations of particle motion. It has been shown that particles drift equations conserve the phase space volume for arbitrary velocity field $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$. When the velocity of the reference frame is fixed to be the plasma fluid velocity defined by the full momentum balance equation (14), the conservation of the adiabatic invariant is shown by direct calcu-

lation. Our principal results are given by the drift kinetic equation (11) and reduced equations (24), (32), and (41) for the evolution of density, energy, and parallel viscosity, respectively.

Our present formulation does not include the finite Larmor radius terms which were considered in Refs. 17, 24, and 26. Such effects are described by the inertial (polarization) and gyroviscosity terms. Such terms can be included into the definition of \mathbf{F} by adding the gyroviscous force $\nabla \cdot \Pi_g$ to the left hand side of Eq. (15). In the latter case, however, we would have to include the higher order inertial terms in the averaged equations of particle motion, which is left for future publication. Nor we consider gyrokinetic equation with the finite and large Larmor radius effects^{33,34} (review of this approach and references can be found in Ref. 34).

Our equations of motion are identical to the slow flow and low plasma pressure limit of the drift kinetic equation obtained in Ref. 26 by the asymptotic averaging of the full Vlasov equation. Our drift kinetic equation in a final form (11) is also similar to the drift kinetic equation for plasma with large flows in Ref. 24, which was derived by a different method. Detailed, term by term, comparison is outside of scope of the present paper. Here, we note only few, most important, points.

The equations in Ref. 24 include the second order drifts responsible for finite Larmor radius (FLR) effects that are neglected in our work. When the latter terms are ignored, the equation for energy evolution in Ref. 24 is identical to our Eq. (8), however, equation for $d\mu/dt$ in Ref. 24 does not have the $mw_{\perp}^2 \mathbf{b} \cdot \nabla \times \mathbf{F} / 2B\omega_c$ term, which is present in our Eq. (9). It was explicitly assumed in Ref. 24 that plasma fluid velocity \mathbf{V} only includes $\mathbf{E} \times \mathbf{B}$ drift and neglects the diamagnetic \mathbf{V}_p and anisotropic pressure \mathbf{V}_{π} contributions. As a result, to the first order in $1/B$, the function \mathbf{F} becomes zero and $mw_{\perp}^2 \mathbf{b} \cdot \nabla \times \mathbf{F} / 2B\omega_c$ drops out from Eq. (9). The work²⁴ has been extended in Ref. 25 to include some effects of anisotropy. However, the reference frame velocity in Ref. 25 only included the $\mathbf{E} \times \mathbf{B}$ drift and flow along the magnetic field (similar approach was also adopted in Ref. 17). This is a principal difference between our work and Refs. 24 and 25. In our approach, the reference frame moves with the total plasma fluid velocity that includes $\mathbf{E} \times \mathbf{B}$, \mathbf{V}_p and \mathbf{V}_{π} drifts. The function \mathbf{F} is also redefined; in our approach this function satisfy the momentum balance equation, cf. our Eqs. (3) and (15) and Eq. (9) of Ref. 25. When diamagnetic \mathbf{V}_p and anisotropic pressure \mathbf{V}_{π} velocities are included in fluid velocity \mathbf{V} , the term $w_{\perp}^2 \mathbf{b} \cdot \nabla \times \mathbf{F} / 2B\omega_c$ in Eq. (9) for $d\mu/dt$ becomes important. It is critically important for phase space volume conservation and for conservation of adiabatic moment as it was discussed in Sec. II C. Such a term does appear in Ref. 25, however, because of the different definition of the function \mathbf{F} , $d\mu/dt \neq 0$ in Ref. 25. We have shown here that for the reference frame velocity \mathbf{V} that includes all the first order terms in $1/B$, i.e., $\mathbf{V} = V_{\parallel} \mathbf{b} + \mathbf{V}_E + \mathbf{V}_p + \mathbf{V}_{\pi}$, the adiabatic invariant defined in the moving reference frame is conserved exactly $d\mu/dt = 0$ to this order. It is worth noting that equation²⁴ can be reliably used to calculate the neoclassical closures for plasmas with large flows if one assumes

$d\mu/dt = 0$ and simply drop this term as it was done, e.g., in Refs. 4, 35, and 36. We note that in Ref. 26, the reference frame velocity was also defined as the complete flow velocity (16), Ref. 26 contains also the discussion of other choice for \mathbf{V} .

We have derived the evolution equations for density, pressure, and parallel viscosity by taking the moments of the drift kinetic equation. The momentum balance equation is used as a definition of the fluid velocity $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$ and has a standard form (14). The density evolution equation is trivially consistent with reduced MHD equations when only the first order drifts $\mathbf{V} = V_{\parallel} \mathbf{b} + \mathbf{V}_E + \mathbf{V}_p + \mathbf{V}_{\pi}$, are included in the velocity. It is shown that the energy evolution equation obtained as the moment of the drift kinetic equation is identical to the reduced MHD equation when the heat flux is determined from the extended (Grad type) equation for the evolution of the heat flux. The extended heat flux, in addition to the standard diamagnetic heat flux by Braginskii \mathbf{q}_{\perp} , contains the \mathbf{q}_{π} , \mathbf{q}_{π^*} , and \mathbf{q}_{χ} contributions related to viscosity tensors Π_{\parallel} , Π_{\parallel}^* , and the isotropic fourth order tensor $\chi \mathbf{I}$. It is important to note that the tensor moments π_{\parallel} , π_{\parallel}^* , and χ form a complete (irreducible) system for the description of the energy flux due to the magnetic gradient drift velocity, $\langle (mw_{\perp}^2 / 2\omega_c)(w_{\perp}^2 / 2 + w_{\parallel}^2) \mathbf{b} \times \nabla \ln B \rangle$, see Eq. (31). Similarly, the tensor moments π_{\parallel} , π_{\parallel}^* , χ , and λ form an irreducible system for the description of the viscosity tensor flux to the magnetic gradient drift velocity, $\langle (mw_{\perp}^2 / 2 - w_{\parallel}^2)(w_{\perp}^2 / 2 + w_{\parallel}^2) \mathbf{b} \times \nabla \ln B / \omega_c \rangle$, see Eq. (40).

Equations (11), (24), (32), and (41) form a basis for designing of neoclassical and Landau damping closures.^{1,5} The preferred approach for closures is the generalized Chapman–Enskog method,^{6,8} which uniquely defines the closures for higher moments in terms of the low order moments. It is worth noting that application of the generalized Chapman–Enskog method for collisionless systems gives the closures which are consistent with the standard closures obtained for strongly collisional plasmas.^{6,8} There are different approaches possible for closure formulation. One can define the plasma density, temperature and velocity as dynamical variables and seek kinetic closures for π_{\parallel} , π_{\parallel}^* , χ , and heat flux q_{\parallel} . Note that in slab geometry π_{\parallel} and q_{\parallel} are responsible for Landau damping effects.⁶ Neoclassical effect are usually described by π_{\parallel} and π_{\parallel}^* ,¹ we note however that the fourth rank isotropic tensor χ in Eq. (32) provides a contribution to the neoclassical energy flux of the same order as π_{\parallel} and π_{\parallel}^* . Alternatively, one can include higher order moments as dynamical variables, e.g., one can use Eq. (41) for π_{\parallel} and seek closures for π_{\parallel} , π_{\parallel}^* , χ , λ and heat fluxes q_{\parallel}^{\perp} and $q_{\parallel}^{\parallel}$. As well, evolution equation for the heat flux can be added.³⁷ It was shown¹⁹ that heat flux evolution equation allows the inclusion of some neoclassical effects into a fluid theory. In toroidal geometry Landau damping and neoclassical effects will generally be coupled in closure relations. Tractable expressions can be obtained by considering contributions only from few toroidal harmonics. This will be reported in a separate publication.

Our reduced equations for density, energy (pressure), and plasma viscosity are equivalent to the two-pressure gyrofluid equations.^{10,38} Similar reduced equations were also

obtained in Ref. 39 from moments of Vlasov equation and in Ref. 26 from moments of the drift kinetic equations. In our derivations, we emphasized the higher order moments which are responsible for the neoclassical fluxes: π_{\parallel} for the particle flux in Eq. (24), π_{\parallel} , π_{\parallel}^* , and χ for the density flux in Eq. (32), and π_{\parallel} , π_{\parallel}^* , χ , and λ for the viscosity flux in Eq. (41). Reduced equations of Ref. 26 also contain the fourth-rank moments, however direct comparison with Ref. 26 is difficult, in part, due to the separation of the contribution of a two-temperature Maxwellian to the fourth-rank moments in Ref. 26.

As it was noted above, our drift kinetic equation and the reduction of extended MHD are done only to the first order in $1/B$. Therefore, our equations contain all diamagnetic and anisotropy effects such as those required in neoclassical theory but they do not contain the second order, $1/B^2$, effects related to finite Larmor radius, and associated Reynolds stress.

It is worth noting that the parallel momentum balance equation (14), which we do not reduce in this paper, also allows simplification by using an expansion in $1/B$ small parameter. This requires the inclusion of the gyroviscosity stress tensor of the first order in $1/B$ (without the FLR effects but related to parallel flow) and is outside of the scope of a present paper. It has been reported recently in Ref. 40.

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APPENDIX A: HEAT FLUX EQUATION

The derivation of the moment hierarchy of Boltzmann kinetic equation is most convenient from the equation in the moving reference frame $f=f(\mathbf{r}, t, \mathbf{w})$, which is written as

$$\begin{aligned} \frac{d}{dt}f + \mathbf{w} \cdot \nabla f + \frac{\partial}{\partial \mathbf{w}} \cdot [(\mathbf{F} - (\mathbf{w} \cdot \nabla)\mathbf{V} + \mathbf{w} \times \omega_c)f] \\ + f \nabla \cdot \mathbf{V} = C, \end{aligned} \quad (\text{A1})$$

where C is a collisional operator, $\omega_c = \omega_c \mathbf{b}$,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla, \quad (\text{A2})$$

and \mathbf{F} is given by Eq. (3). The heat flux evolution equation is obtained by integrating Eq. (A1) with a weighting factor $m w^2 \mathbf{w}/2$. Retaining the relevant term we obtain

$$\begin{aligned} \frac{d}{dt} \left\langle \frac{m w^2}{2} \mathbf{w} \right\rangle + \left\langle \frac{m w^2}{2} \mathbf{w} \right\rangle \nabla \cdot \mathbf{V} \\ + \left\langle [(\mathbf{w} \cdot \nabla)\mathbf{V}] \cdot \frac{\partial}{\partial \mathbf{w}} \frac{m w^2}{2} \mathbf{w} \right\rangle + \nabla \cdot \left\langle \frac{m w^2}{2} \mathbf{w} \mathbf{w} \right\rangle \\ - \mathbf{F} \cdot \left\langle \frac{\partial}{\partial \mathbf{w}} \frac{m w^2}{2} \mathbf{w} \mathbf{w} \right\rangle - \left\langle \mathbf{w} \times \omega_c \cdot \frac{\partial}{\partial \mathbf{w}} \frac{m w^2}{2} \mathbf{w} \right\rangle \\ = \left\langle \frac{m w^2}{2} \mathbf{w} C \right\rangle. \end{aligned} \quad (\text{A3})$$

This first three terms in this equation are responsible for higher order inertial corrections to the heat flux. Such terms are similar to the inertial polarization terms in the density equation due to the inertia and gyroviscosity responsible for inertial (polarization) contribution to the energy evolution equation. The latter effects as well as collisional terms will be neglected here. The remaining terms give

$$\omega_c \mathbf{q} \times \mathbf{b} = \nabla \cdot \mathbf{P} - \mathbf{F} \cdot \left\langle \frac{\partial}{\partial \mathbf{w}} \frac{m w^2}{2} \mathbf{w} \right\rangle, \quad (\text{A4})$$

$$\begin{aligned} \mathbf{P} &\equiv \left\langle \frac{m w^2}{2} \mathbf{w} \mathbf{w} \right\rangle \\ &= \left\langle \frac{T}{m} \left(\frac{w^2}{v_T^2} - \frac{7}{2} \right) m \left(\mathbf{w} \mathbf{w} - \frac{1}{3} w^2 \mathbf{I} \right) \right. \\ &\quad + \frac{7T}{2m} m \left(\mathbf{w} \mathbf{w} - \frac{1}{3} w^2 \mathbf{I} \right) + \frac{2T^2}{3m} \left(\frac{w^4}{v_T^4} - 5 \frac{w^2}{v_T^2} + \frac{15}{4} \right) \mathbf{I} \\ &\quad \left. + \frac{10T^2}{3m} \left(\frac{w^2}{v_T^2} - \frac{3}{4} \right) \right\rangle. \end{aligned} \quad (\text{A5})$$

Introducing the relevant moments, we have

$$\mathbf{P} = \frac{5pT}{2m} \mathbf{I} + \frac{7}{4} v_T^2 \mathbf{\Pi} + \frac{1}{2} v_T^2 \mathbf{\Pi}^* + \frac{1}{6} v_T^2 \chi \mathbf{I}.$$

The second term in Eq. (A4) is transformed as follows:

$$\begin{aligned} \mathbf{F} \cdot \left\langle \frac{\partial}{\partial \mathbf{w}} \frac{m w^2}{2} \mathbf{w} \right\rangle \\ = \mathbf{F} \cdot \left\langle \frac{m w^2}{2} \mathbf{I} + m \mathbf{w} \mathbf{w} \right\rangle \\ = \mathbf{F} \cdot \left\langle m \left(\mathbf{w} \mathbf{w} - \frac{1}{3} w^2 \mathbf{I} \right) + \frac{5m w^2}{6} \mathbf{I} \right\rangle = \mathbf{F} \cdot \mathbf{\Pi} + \frac{5}{2} \mathbf{F} p. \end{aligned} \quad (\text{A6})$$

These result in Eq. (43).

APPENDIX B: SIMPLIFICATIONS OF THE ENERGY BALANCE EQUATION

In this section, we give the details of the simplifications in Eq. (42). Diamagnetic cancellation in Eq. (42) has the form

$$\begin{aligned} \nabla \cdot \left(\frac{3}{2} p \mathbf{V}_p \right) + p \nabla \cdot \mathbf{V}_p + \nabla \cdot \mathbf{q}_\perp \\ = \frac{5}{m \omega_c} \nabla \cdot (pT) \cdot \mathbf{b} \times \nabla \ln B. \end{aligned} \quad (\text{B1})$$

First we simplify in $\mathbf{\Pi}_{\parallel} : \nabla V$ terms containing \mathbf{V}_E

$$\mathbf{\Pi}_{\parallel} : \nabla \mathbf{V}_E = -\frac{1}{2} \pi_{\parallel} \mathbf{V}_E \cdot \nabla \ln B, \quad (\text{B2})$$

and V_{\parallel}

$$\mathbf{\Pi}_{\parallel} : \nabla (V_{\parallel} \mathbf{b}) = \frac{3}{2} \pi_{\parallel} (\mathbf{b} \cdot \nabla V_{\parallel} - \frac{1}{3} \nabla \cdot (\mathbf{b} V_{\parallel})). \quad (\text{B3})$$

It is convenient to separate

$$\mathbf{q}_\pi = \mathbf{q}_\pi^1 + \mathbf{q}_\pi^2, \quad (\text{B4})$$

where

$$\begin{aligned} \mathbf{q}_\pi^1 = & -\frac{5}{2} \frac{T}{m\omega_c} \mathbf{b} \times \nabla \cdot \mathbf{\Pi} + \frac{7}{2\omega_c} \mathbf{b} \times \nabla \cdot \left(\frac{T}{m} \mathbf{\Pi} \right) \\ & - \frac{1}{mn\omega_c} \mathbf{b} \times \nabla p \cdot \mathbf{\Pi}, \end{aligned} \quad (\text{B5})$$

and

$$\mathbf{q}_\pi^2 = -\frac{1}{mn\omega_c} \mathbf{b} \times \nabla \cdot \mathbf{\Pi} \cdot \mathbf{\Pi}. \quad (\text{B6})$$

Nonlinear $\mathbf{\Pi}$ viscous terms have the form

$$\mathbf{\Pi}_\parallel: \nabla \mathbf{V}_\pi = -2\pi_\parallel \mathbf{V}_\pi \cdot \nabla \ln B + \frac{1}{2} \pi_\parallel \mathbf{V}_\pi \cdot \nabla \ln n, \quad (\text{B7})$$

and

$$\begin{aligned} \nabla \cdot \mathbf{q}_\pi^2 = & \frac{\pi_\parallel}{mn\omega_c} \mathbf{b} \times \nabla \ln B \cdot \nabla \pi_\parallel \\ & + \frac{1}{4mn\omega_c} \mathbf{b} \times (\nabla \pi_\parallel - 3\pi_\parallel \nabla \ln B) \cdot \nabla \ln n. \end{aligned} \quad (\text{B8})$$

Using these expressions one can show that

$$\mathbf{\Pi}_\parallel: \nabla \mathbf{V}_\pi + \nabla \cdot \mathbf{q}_\pi^2 = 0. \quad (\text{B9})$$

Further transformations for $\mathbf{\Pi}$ viscous terms are

$$\begin{aligned} \nabla \cdot \left(\frac{3}{2} p \mathbf{V}_\pi \right) + p \nabla \cdot \mathbf{V}_\pi + \mathbf{\Pi}: \nabla \mathbf{V}_p + \nabla \cdot \mathbf{q}_\pi^1 \\ = \frac{7}{4m\omega_c} \nabla (T\pi_\parallel) \cdot \mathbf{b} \times \nabla \ln B. \end{aligned} \quad (\text{B10})$$

The viscous terms containing $\mathbf{\Pi}^*$ become

$$\begin{aligned} \nabla \cdot \mathbf{q}_{\pi^*} = & -\frac{2}{\omega_c} \mathbf{b} \times \nabla \cdot \left(\frac{T}{m} \mathbf{\Pi}^* \right) \cdot \nabla \ln B - \frac{1}{\omega_c} \mathbf{b} \cdot \nabla \\ & \times \nabla \cdot \left(\frac{T}{m} \mathbf{\Pi}^* \right). \end{aligned} \quad (\text{B11})$$

Using these identities,

$$\mathbf{b} \cdot \nabla \times \nabla \cdot \left(\frac{T}{m} \mathbf{\Pi}^* \right) = \frac{3}{2} \mathbf{b} \cdot \nabla \left(\frac{T}{m} \pi_\parallel^* \right) \times \nabla \ln B, \quad (\text{B12})$$

$$\mathbf{b} \times \nabla \cdot \left(\frac{T}{m} \mathbf{\Pi}^* \right) \cdot \nabla \ln B = -\frac{1}{2} \mathbf{b} \times \nabla \left(\frac{T}{m} \pi_\parallel^* \right) \cdot \nabla \ln B, \quad (\text{B13})$$

Eq. (B11) reduces to the form

$$\nabla \cdot \mathbf{q}_{\pi^*} = -\frac{1}{2\omega_c} \mathbf{b} \times \nabla \left(\frac{T}{m} \pi_\parallel^* \right) \cdot \nabla \ln B. \quad (\text{B14})$$

The term with higher order scalar moment χ is

$$\nabla \cdot \mathbf{q}_\chi = -\frac{2}{3\omega_c} \mathbf{b} \times \nabla \left(\frac{T}{m} \chi \right) \cdot \nabla \ln B. \quad (\text{B15})$$

Assembling Eqs. (B1)–(B3), (B10), (B14), and (B15) in Eq. (42), one obtains Eq. (32).

APPENDIX C: SOME USEFUL IDENTITIES

The following relations are valid in the low pressure approximation so that $\nabla \times \mathbf{B} = \mathbf{0}$ and $(\mathbf{b} \cdot \nabla) \mathbf{b} = \nabla_\perp \ln B$. From the definitions of \mathbf{V}_E and \mathbf{V}_p , one finds

$$\nabla \cdot \mathbf{V}_E = -2\mathbf{V}_E \cdot \nabla \ln B, \quad (\text{C1})$$

$$\nabla \cdot \mathbf{V}_p = -2\mathbf{V}_p \cdot \nabla \ln B + \mathbf{V}_p \cdot \nabla \ln n. \quad (\text{C2})$$

The parallel viscosity is defined by Eq. (18) so that

$$\nabla \cdot \mathbf{\Pi} = \frac{3}{2} \mathbf{b} (\mathbf{b} \cdot \nabla \pi_\parallel) + \frac{3}{2} \pi_\parallel (\mathbf{b} \nabla \cdot \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{b}) - \frac{1}{2} \nabla \pi_\parallel, \quad (\text{C3})$$

and the diamagnetic velocity due to pressure anisotropy takes the form

$$\mathbf{V}_\pi = -\frac{1}{2} \frac{1}{mn\omega_c} \mathbf{b} \times \nabla \pi_\parallel + \frac{3}{2} \frac{\pi_\parallel}{mn\omega_c} \mathbf{b} \times \nabla \ln B, \quad (\text{C4})$$

and

$$\nabla \cdot \mathbf{V}_\pi = -\mathbf{V}_\pi \cdot \nabla \ln n + \mathbf{V}_\pi \cdot \nabla \ln B. \quad (\text{C5})$$

The definition of parallel viscosity gives the following relation for arbitrary \mathbf{V} :

$$\mathbf{\Pi}: \nabla \mathbf{V} = \frac{3}{2} \pi_\parallel (\mathbf{b} \mathbf{b}: \nabla \mathbf{V} - \frac{1}{3} \nabla \cdot \mathbf{V}). \quad (\text{C6})$$

The first term on the right hand side here can be put in the form

$$\mathbf{b} \mathbf{b}: \nabla \mathbf{V} = b \cdot \nabla V_\parallel - \mathbf{V}_\perp \cdot (\mathbf{b} \cdot \nabla) \mathbf{b} = b \cdot \nabla V_\parallel - \mathbf{V}_\perp \cdot \nabla \ln B. \quad (\text{C7})$$

From the definition of \mathbf{F} in Eq. (15), it follows

$$\frac{1}{\omega_c} \mathbf{b} \cdot \nabla \times \mathbf{F} = \mathbf{V}_p \cdot \nabla \ln n + \mathbf{V}_\pi \cdot \nabla \ln n - 3\mathbf{V}_\pi \cdot \nabla \ln B. \quad (\text{C8})$$

The following identities are also used:

$$\mathbf{b} \cdot \nabla \times \nabla \cdot \mathbf{\Pi} = \frac{3}{2} \mathbf{b} \cdot \nabla \pi_\parallel \times \nabla \ln B, \quad (\text{C9})$$

$$(\mathbf{b} \times \nabla \cdot \mathbf{\Pi}) \cdot \nabla \ln B = -\frac{1}{2} \mathbf{b} \times \nabla \pi_\parallel \cdot \nabla \ln B. \quad (\text{C10})$$

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