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Saturation of magnetic islands in equilibria with a finite current gradient. Part I: asymptotic theory

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Abstract
The problem of saturation of magnetic islands for the case of a finite current gradient at the rational surface has been revisited. The asymptotic procedure is presented giving the explicit expressions for the perturbed magnetic flux function at saturation. The resulting nonlinear equation for the island width has been obtained in the form similar to the previous work but with fully analytical expressions for the asymptotic series.

Keywords: magnetic islands, tearing modes, reconnection

(Some figures may appear in colour only in the online journal)

1. Introduction

Dynamics of magnetic islands is one of the fundamental problems of general magnetohydrodynamic theory with numerous applications in tokamak and space physics. Over the years analytical theory that has been developed to study magnetic islands in various situations, provided physics insight into the problem and served as a guidance for numerical simulations. Such analytical theory is primarily based on the asymptotic matching procedure, which relies on the scale separation between the inner (nonlinear) and outer (linear) regions. Within this setting, the magnetic island growth was studied in pioneering work by Rutherford [1]. Subsequently, the saturation of magnetic islands was investigated with different variations of the asymptotic matching techniques [2–11]. This approach results in the Rutherford type equation with additional terms due to the gradients of the equilibrium current profile. This equation can be complemented with extra terms describing the localized heating, current drive and external magnetic perturbations. Such modified Rutherford equation forms the basis of the current approach for the Resonant Magnetic Perturbations (RMP) control [12, 13] and suppression of Neoclassical Tearing Modes (NTM) [14–18]. Precise details of the magnetic flux structure are important in practical realization of NTM the control schemes [19, 20]. The goal of this paper is to present the the asymptotic procedure describing the deformation of the magnetic flux function of finite width magnetic islands in the case of the equilibrium current profile with a finite current gradient at the rational surface. As a result we are able to obtain the closed form integral expressions for all relevant terms.

The case of the symmetric current profiles was considered earlier in [6, 8], see also [21]. In this paper, we consider the magnetic islands in the equilibrium with a finite current gradient at the rational surface. The finite current gradient effects appear in the Rutherford equation in the second order, i.e. as w/a^2 terms [2, 5], where w is an island width and $a = J_0/J_0'$ is the current gradient length. Additional terms appear in combination with the asymmetry induced by the external boundary conditions described by the parameter Σ', i.e. as $w\Sigma'/a$ terms. The structure of our nonlinear equation for the island width is similar to that of the previous work [7, 9–11] The closed form integral expressions for numerical coefficients obtained in our paper give the numerical value which is close to that in the previous work

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[7, 9–11]. The results of the analytical theory are compared with numerical simulations in companion paper, Part II.

2. Basic equations

We consider a magnetic equilibrium in the geometry of the magnetic island based on standard magnetohydrodynamic equations in 2-D geometry with a constant guiding magnetic field along the z-direction [22]. Neglecting the inertial effects, the current closure equation gives

$$\nabla_{\|} J = 0, \tag{1}$$

where $\nabla_{\|}$ is the parallel gradient operator along the total transverse magnetic field, $\nabla_{\|} = \mathbf{B}\cdot\nabla/|\mathbf{B}|$, where the magnetic field is characterized by the $\mathbf{z}$ component of the magnetic vector potential $\mathbf{B} = \nabla\times(A_z\hat{\mathbf{z}})$. It is convenient to use $\psi = -A_z$, so that $\mathbf{B}(x, y, t) = \hat{\mathbf{z}}\times\nabla\psi$ and the Ampere' law for the $\mathbf{z}$ component of the current takes the form

$$J = \nabla_{\perp}^{2}\psi, \tag{2}$$

where $\nabla_{\perp}^{2} = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The transverse magnetic field consists of the equilibrium and perturbed parts: $\mathbf{B}(x, y)\,\mathbf{B}_0(y) + \widetilde{\mathbf{B}}(x, y, t)$. The equilibrium (sheared) magnetic field $\mathbf{B}_0 = B_0(x)\hat{\mathbf{y}} = \partial\psi_0(x)/\partial x$, has a singularity at $x = 0$, $B_0(0) = 0$. Note that the perturbation is independent of z direction.

It is convenient to define the total magnetic flux as a sum of the equilibrium $\psi = \psi_0(x)$ and the perturbation $\widetilde{\psi} = \widetilde{\psi}(x, y, t)$:

$$\psi(x, \xi) = \psi_0 - \widetilde{\psi}\,. \tag{3}$$

The parallel gradient operator can be expressed in the form $\nabla_{\|} = \mathbf{B}\cdot\nabla/|\mathbf{B}| = |\mathbf{B}|^{-1}\hat{\mathbf{z}}\cdot\nabla\psi\times\nabla$. The current closure equation (1) then becomes $\hat{\mathbf{z}}\cdot\nabla\psi\times\nabla J = 0$ giving the current in the form of the magnetic flux function

$$J = J(\psi)\,. \tag{4}$$

The current is also constrained by Ohm's law in the form

$$-\nabla_{\|}\phi + E_0 = \eta J, \tag{5}$$

where E_0 is the equilibrium electric field that satisfies the equation

$$E_0 = \eta J_0, \tag{6}$$

and ϕ is the perturbed electrostatic potential.

In our paper, we consider two cases: in the first model (model A) we assume that plasma resistivity is uniform in radial direction, η = const and $E_0 = E_0(x)$, $J_0 = \nabla_{\perp}^2\psi_0(x)$. In alternative model [10, 11] (model B) one assumes that the electric field is constant but plasma resistivity is non-uniform, E_0 = const, $\eta = \eta(x)$. We treat this case in the appendix E.

We consider the perturbation periodic in y. It is convenient to use the dimensionless variable $\xi = ky$, where k is the wave vector in y (poloidal) direction. It is useful to transform the gradient operator along the total magnetic field to new variables: $(x, \xi = ky) \to (\psi, \xi)$:

$$\nabla_{\|} = B^{-1}\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right) = kB^{-1}\frac{\partial\psi}{\partial x}\left(\frac{\partial}{\partial\xi}\right)_{\psi}, \tag{7}$$

where the derivative $(\partial/\partial\xi)_\psi$ is taken at constant ψ. In (ψ, ξ) variables, equation (5) can be written

$$-B^{-1}\frac{\partial\psi}{\partial x}\left(\frac{\partial\phi}{\partial\xi}\right)_{\psi} + E_0 = \eta J\,. \tag{8}$$

Uniqueness of the solution for ϕ in periodic variable in ξ and equations (4) and (6) give the the solubility condition in the form

$$\begin{aligned}\langle E_0(x)\,(\partial\psi/\partial x)^{-1}\rangle &= \eta\,\langle J(\psi)\,(\partial\psi/\partial x)^{-1}\rangle\\ &= \eta J(\psi)\,\langle(\partial\psi/\partial x)^{-1}\rangle\,,\end{aligned} \tag{9}$$

where the angle brackets simply mean the integral over periodic variable $\langle(...)\rangle = \oint(...)\mathrm{d}\xi$, which is done at constant ψ.

One then finds the expression for the total current in the form

$$J(\psi) = \frac{\langle J_0(x)\,(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\,. \tag{10}$$

For the perturbed current one has $\widetilde{J} = J(\psi) - J_0 = -\nabla_{\perp}^2\widetilde{\psi}$ so that the final equation is

$$-\nabla_{\perp}^2\widetilde{\psi} = \frac{\langle J_0(x)\,(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle} - J_0(x)\,. \tag{11}$$

This is the implicit nonlinear integral-differential equation that describes the equilibria with magnetic island.

3. Nonlinear equation in the inner region

We assume generic current profile which has a finite current gradient at the rational surface, $J_0' \neq 0$. The second derivative of the current is also included. We assume that the island width is small compared to the characteristic length scale of the equilibrium current, $w \ll a$, so that the current in the nonlinear region can be approximated as

$$J(x) = J_0(0) + J_0'x + J_0''\frac{x^2}{2}, \tag{12}$$

or, respectively,

$$\psi_0 = \psi_0''\frac{x^2}{2} + \psi_0^{\mathrm{III}}\frac{x^3}{6} + \psi_0^{\mathrm{IV}}\frac{x^4}{24}, \tag{13}$$

where $J_0(x) = \psi_0''$ was used.

The basic nonlinear equation (11) then takes the form

$$\begin{aligned}-\nabla_{\perp}^2\widetilde{\psi} &= \psi_0^{\mathrm{III}}\left[\frac{\langle x(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle} - x\right]\\ &+ \frac{\psi_0^{\mathrm{IV}}}{2}\left[\frac{\langle x^2(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle} - x^2\right].\end{aligned} \tag{14}$$

It is convenient to represent the solution as the sum of the 'constant ψ approximation' term: $\psi_1\cos\xi$ and an additional function $H(x, \xi)$, so that

$$\widetilde{\psi} = \psi_1[\cos\xi + H(x, \xi)]\,, \tag{15}$$

where $\xi = ky$. Our basic nonlinear equation (14), written in term of the function $H(x, \xi)$, becomes

$$\frac{\partial^2}{\partial x^2}H(x,\xi) = (k^2 - b^{-2})\cos\xi - b^{-2}H(x,\xi) - \frac{\partial^2 H(x,\xi)}{\partial y^2} - \frac{4}{w^2 a}\left(\frac{\langle x(\partial\psi\partial x)^{-1}\rangle}{\langle(\partial\psi\partial x)^{-1}\rangle} - x\right) + b^{-2}\frac{\langle(\cos\xi + H(x,\xi))(\partial\psi\partial x)^{-1}\rangle}{\langle(\partial\psi\partial x)^{-1}\rangle}, \quad (16)$$

where $a = J_0/J_0{'} = \psi_0{''}/\psi_0{'''}$, $b^2 = -J_0/J_0{''} = -\psi_0{''}/\psi_0^{\mathrm{IV}}$ and the magnetic island half-width w is defined by the relation $w^2 = 4\psi_1/\psi_0{''}$.

4. Perturbative solution in the nonlinear region

Nonlinear equation (16) involves averaging over the magnetic flux surfaces that makes this equation nonlocal and implicit, since the averaging is done over the magnetic surfaces, which are determined by the unknown function $\psi = \psi(x,\xi)$

$$\psi = \frac{1}{2}\psi_0{''}x^2 + \frac{1}{6}\psi_0{'''}x^3 + \psi_0^{\mathrm{IV}}\frac{x^4}{24} - \psi_1(\cos\xi + H(x,\xi)). \quad (17)$$

The dimensionless magnetic flux function $u = \psi/\psi_1$ is defined as

$$u = u_0(x,\xi) + u_1(x,\xi) = 2\frac{x^2}{w^2}\left(1 + \frac{x}{3a}\right) - \cos\xi - H, \quad (18)$$

where $u_0(x,\xi) = 2x^2/w^2 - \cos\xi$ and $u_1(x,\xi) = 2x^3/3w^2a - H(x,\xi)$. Note that in our ordering the term $\psi_0^{\mathrm{IV}}x^4$ in (13) and (18) can be neglected, but it will be retained in equation (16).

Nonlinear equation (16) can be solved by the expansion using the small parameter $H(x,\xi) \ll 1$. The function $x = x(\psi,\xi)$, which determines the magnetic flux surfaces, is sought in the series form $x = x_0(u,\xi) + x_1(u,\xi) + \ldots$, where $x_0(u,\xi)$ is the lowest order (constant ψ) solution

$$x_0^2 = \frac{2}{\psi_0{''}}(\psi + \psi_1\cos\xi) = \frac{w^2}{2}(u + \cos\xi), \quad (19)$$

and $x_1(u,\xi)$ is the next order term

$$x_1(u,\xi) = -\frac{1}{6}\frac{x_0^2}{a} + \frac{w^2 H(x_0,\xi)}{4x_0}. \quad (20)$$

Averaging of the magnetic surfaces in (16) requires the volume element between magnetic surfaces characterized by the value of the derivative $\partial\psi/\partial x$. Deformation of the magnetic surfaces due to deviation from $\psi = const$ modifies this volume element as follows

$$\frac{\partial\psi}{\partial x} = \psi_0{''}x_0(1 + M), \quad (21)$$

where to the first order

$$M(u,\xi) = \frac{1}{3}\frac{x_0}{a} - \frac{w^2}{4}\frac{\partial}{\partial x_0}\frac{H(x_0,\xi)}{x_0}. \quad (22)$$

Only the first order expression for M will be required in our ordering.

Taking into account equations (21) and (16) becomes

$$\frac{\partial^2}{\partial x^2}H(x,\xi) = (k^2 - b^{-2})\cos\xi - b^{-2}H(x,\xi) - \frac{\partial^2 H(x,\xi)}{\partial y^2} - \frac{4}{w^2 a}\left[\frac{\langle(1+M)^{-1}(1 + x_1/x_0)\rangle}{\langle x_0^{-1}(1+M)^{-1}\rangle} - x\right] + b^{-2}\frac{\langle x_0^{-1}\cos\xi\rangle}{\langle x_0^{-1}\rangle}. \quad (23)$$

Expanding in small parameters x_1/x_0, H, and M, the right hand side of (23) can be represented in the form

$$\frac{\partial^2}{\partial x^2}H(x,\xi) = F(x,\xi) + (k^2 - b^{-2})\cos\xi - b^{-2}H(x,\xi) - \frac{\partial^2 H(x,\xi)}{\partial y^2}, \quad (24)$$

where

$$F(x,\xi) = \frac{4x}{w^2 a} + F_0(u) + F_{1G}(u) + F_{1M}(u) + F_{1b}(u), \quad (25)$$

and respective components of $F(x,\ \xi)$ are given by the expressions:

$$F_0(u) = -\frac{4}{w^2 a}\,\mathrm{sgn}(x)\frac{\langle 1\rangle}{\langle x_0^{-1}\rangle}\sigma(u-1), \quad (26)$$

$$F_{1G}(u) = -\frac{4}{w^2 a}\frac{1}{\langle x_0^{-1}\rangle}\langle G\rangle, \quad (27)$$

$$F_{1M}(u) = -\frac{4}{w^2 a}\frac{\langle x_0^{-1}M\rangle\langle 1\rangle}{\langle x_0^{-1}\rangle^2}, \quad (28)$$

$$F_{1b}(u) = b^{-2}\frac{\langle x_0^{-1}\cos\xi\rangle}{\langle x_0^{-1}\rangle}. \quad (29)$$

Here G is defined by the expression

$$G = -\frac{1}{2}\frac{x_0}{a} + \frac{w^2}{4}\frac{H_0{'}}{x_0}, \quad (30)$$

and M is given by equation (22).

It is instructive to establish that the nonlinear equation (24) in the linear limit $x/w \gg 1$ reduces to the linear equation in the outer region. The equation for the outer linear region is discussed in appendix B. The full expressions for the nonlinear current functions $F_0(u)$, $F_{1G}(u)$, $F_{1M}(u)$, and $F_{1b}(u)$ in equation (24) and their linear asymptotics are described in appendix C, where it is also shown that the linear limit of equation (24) coincides with the linear equation in the outer region. This guarantees that the nonlinear solution of equation (23) will match with the linear solution in the outer region.

General solution of (24) can be written in the form

$$H(x,\xi) = \int^x \mathrm{d}x' \int^{x'} F(x'',\xi)\,\mathrm{d}x''. \quad (31)$$

By changing the order of integration we have

$$H(x,\xi) = x\int_{c_2(\xi)}^x F(x',\xi)\,\mathrm{d}x' - \int_{c_1(\xi)}^x x'F(x',\xi)\,\mathrm{d}x'. \quad (32)$$

In principle, there are two arbitrary integration constants in this solution: $c_1(\xi)$ and $c_2(\xi)$. The coefficient $c_1(\xi)$ defines the value of $H(x, \xi)$ at $x = 0$. This correspond to an arbitrary constant in the magnetic flux function. Thus one can set $c_1(\xi) = 0$ without resricting the general solution. The $c_2(\xi)$ coefficient remains finite and has to be determined by matching with the outer (linear) region.

It is useful to explicitly establish ordering of various terms in equation (24). From equation (26), one finds that the leading order nonlinear current is

$$F_0 \sim O\left(\frac{1}{wa}\right). \tag{33}$$

Then, the lowest order solution of equation (24), $H_0(x, \xi)$ is

$$H_0 \sim O\left(\frac{w}{a}\right), \tag{34}$$

and equation (22) gives

$$M_0 \sim O\left(\frac{w}{a}\right). \tag{35}$$

The nonlinear current functions $F_{1M}(u)$, $F_{1G}(u)$, and $F_{1b}(u)$ are of the next order:

$$F_{1G}(u) \sim F_{1M} \sim H_0F_0 \sim H_0M_0 \sim O\left(\frac{1}{a^2}\right), \tag{36}$$

$$F_{1b} \sim O\left(\frac{1}{b^2}\right). \tag{37}$$

We assume that $a \sim b$ and thus $F_{1G}(u) \sim F_{1M} \sim F_{1b}$. Therefore, from (24) it follows that the next order solution of (24): $H_1 \sim O(w^2/a^2) \sim O(w^2/b^2)$.

The matching with the linear solution in the outer region is performed by expanding the nonlinear solution in the series of the $\cos\xi$ harmonics: $H\ (x\ \xi) = \mathrm{h}^0\ (x) + h^{(1)}\ (x) \cos\xi + h^2 \cos 2\xi + \ldots$. The equation for the nonlinear island width follows from the matching of the first harmonic $h\ (x)$ with the linear solution in the outer region. In what follows, we drop the superscript, $h^{1\}}(x) \to h(x)$. It is convenient to work with the first derivative of the matching function determined by the equation

$$\frac{\mathrm{d}h\,(x)}{\mathrm{d}x} = \frac{2}{\pi}\int_0^{\pi} \cos\,\xi \frac{\partial H\,(x,\xi)}{\partial x}\,(x,\xi)\,, \tag{38}$$

where $H(x, \xi)$ is found from the equation (24)

$$\frac{\partial H\,(x,\xi)}{\partial x} = \int_0^{x} F\,(x\,)\,\mathrm{d}x = \int_0^{u(x,\xi)} F\,(u,\xi)\,\frac{\mathrm{d}u}{\mathrm{d}u\,/\,\mathrm{d}x}\,. \tag{39}$$

The integration in (39) is performed by the transformation to the $u = u(x, \xi)$ variable, where the Jacobian of the transformation is determined by equation (21): $\mathrm{d}u\,/\,\mathrm{d}x = (2\sqrt{2}\,/\,w)\sqrt{u + \cos\xi}\,(1 + M)$. Then, the derivative of the matching function $h(x)$ is defined by the following expression:

$$\frac{\mathrm{d}h\,(x)}{\mathrm{d}x} = \frac{\mathrm{d}h_0\,(x)}{\mathrm{d}x} + \frac{\mathrm{d}h_{1F_0}\,(x)}{\mathrm{d}x} + \frac{\mathrm{d}h_{1G}}{\mathrm{d}x} + \frac{\mathrm{d}h_{1M}}{\mathrm{d}x} + \frac{\mathrm{d}h_{1FM}}{\mathrm{d}x} + \frac{\mathrm{d}h_{1b}}{\mathrm{d}x}\,. \tag{40}$$

The derivative of the lowest order function $h_0(x) \sim w/a$ is given by

$$\frac{\mathrm{d}h_0}{\mathrm{d}x} = \frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\int_{-\cos\xi}^{u_0(x,\xi)} F_0\,(u)\;\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}\,. \tag{41}$$

We also need a full expression for $h_0(x)$, which will be discussed below. The h_{1F_0}, h_{1G}, h_{1M}, h_{1FM}, and h_{1b} are the first order functions, $h_1 \sim w^2/a^2$. The h_{1G}, h_{1M}, and h_{1b} are directly generated by the first order functions $F_{1G}\ (u)$, $F_{1M}\ (u)$ and $F_{1b}\ (u)$, respectively,

$$\frac{\mathrm{d}h_{1G}}{\mathrm{d}x} = \frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\int_{-\cos\xi}^{u_0(x,\xi)} F_{1G}\,(u)\;\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}, \tag{42}$$

$$\frac{\mathrm{d}h_{1M}}{\mathrm{d}x} = \frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\int_{-\cos\xi}^{u_0(x,\xi)} F_{1M}\,(u)\;\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}, \tag{43}$$

$$\frac{\mathrm{d}h_{1b}}{\mathrm{d}x} = \frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\int_{-\cos\xi}^{u_0(x,\xi)} F_{1b}\,(u)\;\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}\,\}}\,. \tag{44}$$

The variable transformation in (39) produces two additional terms of the first order: h_{1FM} and $h_{1F0.}$ The h_{1FM} is generated from the lowest order function $F_0(u)$ due to the additional term M in the variable transformation in (39)

$$\frac{\mathrm{d}h_{1FM}}{\mathrm{d}x} = -\frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\int_{-\cos\xi}^{u_0(x,\xi)} F_0(u)\,M(u,\xi)\;\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}, \tag{45}$$

and h_{1F_0} appears from $F_0(u)$ as a result of the first order correction in the upper limit of the integration in (39)

$$\frac{\mathrm{d}h_{1F_0}}{\mathrm{d}x} = \frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\int_{u_0(x,\xi)}^{u_0(x,\xi)+u_1(x,\xi)} F_0\,(u)\;\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}$$
$$\simeq \frac{w}{\sqrt{2}\,\pi}\int_0^{\pi}\cos\xi\,\mathrm{d}\xi\,\frac{F_0\,(u)}{\sqrt{u+\cos\xi}}\Bigg|_{u=u_0(x,\xi)} u_1\,(x,\xi)\,. \tag{46}$$

The expressions for the matching functions $\mathrm{d}h_{1FM}/\mathrm{d}x$ and $\mathrm{d}h_{1F_0}\,/\,\mathrm{d}x$ can alternatively be derived from the integral $\int_0^x F_0\,(u)\,\mathrm{d}x$ via the expansion of the $u(x, \xi)$ function in the argument of $F_0(u)$. Details of this derivation are given in appendix F.

5. Nonlinear solution in the leading order

5.1. Full expression for $H_0(x, \xi)$ function

In our iterative procedure, calculations of the first order expressions F_{1G}, F_{1M} as well as h_{1FM} in equations (42), (43) and (45) require the knowledge of the leading order solution $H_0(x, \xi)$ that enters the expressions for G and M in equations (26)–(28). From (24), equation for $H_0(x, \xi)$ takes the form

$$\frac{\partial^2}{\partial x^2}H_0\,(x,\xi) = F_0\,(u) = \frac{4x}{w^2a} - \frac{4}{w^2a}\,\mathrm{sgn}\;(x)\,\frac{\langle 1\rangle\,\sigma\,(u-1)}{\langle x_0^{-1}\rangle}\,. \tag{47}$$

Here $\sigma(u-1)$ is Heaviside function, $\sigma(u-1)=1$ for $u>1$, $\sigma(u-1)=0$, $u<1$. The flux functions $\langle x_0^{-1}\rangle$, <1> and $\beta(u)$ are defined in appendix A. Equation (47) can be integrated directly by using transformation to the u variable (A.5) in the leading order:

$$\int_0^x \mathrm{d}x = \frac{w}{2\sqrt{2}}\int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}, \tag{48}$$

so that one obtains from (47)

$$\frac{\partial H_0(x,\xi)}{\partial x} = \frac{2x^2}{w^2 a} - \frac{\pi}{a}\,\mathrm{sgn}\,(x)\int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{\sigma(u-1)\,\mathrm{d}u}{\beta(u)\sqrt{u+\cos\xi}} + \frac{s}{a}\cos\xi. \tag{49}$$

Integrating it one more time one has

$$H_0(x,\xi) = \frac{2x^3}{3w^2a} - \frac{\pi w}{2\sqrt{2}\,a}\,\mathrm{sgn}\,(x)\int_{-\cos\xi}^{2x^2/w^2-\cos\xi} \times\frac{\mathrm{d}u'}{\sqrt{u'+\cos\xi}}\int_{-\cos\xi}^{u'}\frac{\sigma(u''-1)\,\mathrm{d}u''}{\beta(u'')\sqrt{u''+\cos\xi}} + s\frac{x}{a}\cos\xi. \tag{50}$$

Here, s is yet arbitrary integration constant that will be defined later by matching with external solution, which also dictates the choice of the $\cos\xi$ function in this term. The order of integration in (50) can be reversed

$$\int_{-\cos\xi}^{u}\mathrm{d}u'\int_{-\cos\xi}^{u'}\mathrm{d}u'' = \int_{-\cos\xi}^{u}\mathrm{d}u''\int_{u''}^{u}\mathrm{d}u', \tag{51}$$

and one integration in (50) can be completed in the closed form resulting in the expression

$$H_0(x,\xi) = \frac{2x^3}{3w^2a} - \frac{\pi x}{a}\,\mathrm{sgn}\,(x)\int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{\sigma(u'-1)\,\mathrm{d}u'}{\beta(u')\sqrt{u'+\cos\xi}} + \frac{\pi w}{\sqrt{2}\,a}\,\mathrm{sgn}\,(x)\int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{\sigma(u'-1)\,\mathrm{d}u'}{\beta(u')} + s\frac{x}{a}\cos\xi. \tag{52}$$

Alternatively, this solution directly follows from (32) in the form $H_0(x,\xi)=x\int F(x)\,\mathrm{d}x-\int x\mathrm{d}x$ and using (A.4) for $x=x(u,\xi)$. The lower limit of the integrals in (52) can be set at $u=1$ because the function $<1>/<x_0^{-1}>$ is zero inside the magnetic separatrix.

The function M, which is the Jacobian of the transformation $(x,\xi)\to(x_0,\xi)$, describes the deformation of the magnetic flux surfaces. From (22) and (52) it follows that in zeroth order, the functions M and G are defined by the expressions

$$M = \frac{1}{3}\frac{x_0}{a} - \frac{w^2}{4}\frac{\partial}{\partial x_0}\frac{H(x_0,\xi)}{x_0} = \frac{w^3\pi}{4x_0^2\sqrt{2}\,a}\times\mathrm{sgn}\,(x)\int_{-\cos\xi}^{u}\frac{\sigma(u'-1)\mathrm{d}u'}{\beta(u')}, \tag{53}$$

$$G = -\frac{1}{2}\frac{x_0}{a} + \frac{w^2}{4}\frac{H_0'}{x_0} = -\frac{w\sqrt{2}\,\pi}{4a}\frac{1}{\sqrt{u+\cos\xi}} \times\mathrm{sgn}\,(x)\int_1^u\frac{\mathrm{d}u'}{\beta(u')\sqrt{u'+\cos\xi}} + \frac{\sqrt{2}\,ws}{4a}\frac{\cos\xi}{\sqrt{u+\cos\xi}}. \tag{54}$$

Here, the lower integration limit has been changed to $u=1$. Note that $M(x=0)=0$ in agreement with equation (22); recall that we set $H=0$ for $x=0$.

5.2. Matching of the first harmonic in the leading order

The nonlinear function $H_0(x,\xi)$ found in the previous section, contains the integration constant that has to be determined by matching with external region. The first harmonic of the inner (nonlinear) solution is found from $H(x,\xi)$ in the form

$$h(x) = \frac{2}{\pi}\int_0^{\pi}\cos\,\xi H(x,\xi). \tag{55}$$

In general, the function $H(x,\ \xi)$ has mixed parity. Respectively, function $h(x)$ can be represented as a sum of the even (symmetric), h^s, and odd (anti-symmetric) h^aparts, $h(x)=h^a(x)+h^s(x)$. The leading order solution H_0 defines the anti-symmetric part of $h(x)$:

$$h^a(x) = \frac{2}{\pi}\int_0^{\pi}\cos\,\xi\mathrm{d}\xi\,H_0(x,\xi), \tag{56}$$

which gives

$$h^a(x) = -\frac{2x}{a}\int_0^{\pi}\mathrm{d}\xi\cos\xi\int_1^{2x^2/w^2-\cos\xi}\frac{\sigma(u'-1)\,\mathrm{d}u'}{\beta(u')\sqrt{u'+\cos\xi}} + \frac{\sqrt{2}\,w}{a}\int_0^{\pi}\mathrm{d}\xi\cos\xi\int_1^{2x^2/w^2-\cos\xi}\frac{\sigma(u'-1)\,\mathrm{d}u'}{\beta(u')} + s\frac{x}{a}. \tag{57}$$

This expression can be simplified by changing the order of integration as follows:

$$\int_0^{\pi}\mathrm{d}\xi\int_1^{2x^2/w^2-\cos\xi}\mathrm{d}u = \int_1^{2x^2/w^2+1}\mathrm{d}u\int_{\xi_{m'}}^{\pi}\mathrm{d}\xi. \tag{58}$$

Here

$$\cos\xi_{m'} = 2\frac{x^2}{w^2} - u, \tag{59}$$

so that

$$\xi_{m'} = \arccos\left(2\frac{x^2}{w^2}-u\right)\ \text{for}\ 2\frac{x^2}{w^2}-1<u<2\frac{x^2}{w^2}+1, \tag{60}$$

and

$$\xi_{m'} = 0\ \text{for}\ u<2\frac{x^2}{w^2}-1. \tag{61}$$

Then, one obtains for (57)

$$h^a = -\frac{2x}{a}\int_1^{2x^2/w^2+1}\frac{\sigma(u'-1)\,\mathrm{d}u'}{\beta(u')}\int_{\xi_{m'}}^{\pi}\mathrm{d}\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}} + \frac{\sqrt{2}\,w}{a}\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{\sigma(u'-1)\,\mathrm{d}u'}{\beta(u')}\int_{\xi_{m'}}^{\pi}\mathrm{d}\xi\cos\xi + s\frac{x}{a}. \tag{62}$$

For large $x\gg w$, outside of the magnetic island, this expression has the following asymptotic calculated in (D.13):

$$h_0 = \frac{x}{a}\ln\left(\frac{x}{w}\right) + \frac{x}{a}\left(\frac{\ln 2}{2} - 2g_0\right) + s\frac{x}{a}. \tag{63}$$

This expression is to be matched to the odd part of the outer solution, which is given by the expression (B.9):

$$h^a(x) \to \left[\frac{x}{a}\ln\left|\frac{x}{a}\right| + \frac{\Sigma'}{2}x\right], \tag{64}$$

where it has been assumed $\psi_{\text{out}} = \psi_1$. Matching (63) and (64) determines the s coefficient

$$s = \frac{a\Sigma'}{2} + 2g_0 + \ln\left(\frac{w}{\sqrt{2}\,a}\right). \tag{65}$$

This is the main result of the matching of the anti-symmetric part of the solution. The island saturation equation is obtained from the next order, symmetric part of $H_1(x, \xi)$, which is found from equations (42)–(46).

The Σ' parameter in the outer solution can be redefined by changing the form of the logarithmic term [7, 9–11], e.g. one can write the outer solution as

$$\psi_{\text{out}}\left[\frac{x}{a}\ln\left(\frac{x}{a}\right) + \frac{\Sigma'}{2}x\right] = \psi_{\text{out}}\left[\frac{x}{a}\ln\left(\frac{x}{L}\right) + \frac{\Sigma_R'}{2}x\right], \tag{66}$$

where

$$\Sigma_R' = \Sigma' + \frac{2}{a}\ln\left(\frac{L}{a}\right), \tag{67}$$

and L is some other normalization length. Obviously, matching of the inner solution (63) with (66) gives the same expression for $s_R = s$ and the solution in nonlinear regions remains the same.

6. The matching of the first order solution and final equation for magnetic island

Nonlinear dispersion equation which determines the island width at saturation is found from the matching of the outer and inner solutions in the first order. We work with the derivative of the eigen-function $\mathrm{d}h(x)/\mathrm{d}x$. The asymptotics of the full solution in the outer region from (B.5) has the form

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{\Sigma'}{2} + \frac{1}{a}\ln\left|\frac{x}{a}\right| + \frac{1}{a} - \frac{3}{2}\frac{x}{a^2} \pm \frac{\Delta'}{2}, \tag{68}$$

for $x > 0$ and $x < 0$, respectively. Note that this solution was normalized to unity at the origin, $\psi(0) = 1$. The first three terms in this expression correspond to the the anti-symmetric part which was matched with the lowest order solution $h^a(x)$ (Section V.B). The fourth and fifth terms in (68) are to be matched to the inner first order solution. The fourth term is in fact matched identically to the inner solution; it does not provide any new information and is only useful to confirm the correct form of the nonlinear solution. It is the the matching of the fifth term that gives the equation for the magnetic island width.

The derivatives of the nonlinear solution $\mathrm{d}h_1/\mathrm{d}x$ are calculated from equations (38) and (40). Full expressions for the corresponding functions $\mathrm{d}h_{1G}/\mathrm{d}x$, $\mathrm{d}h_{1M}/\mathrm{d}x$, $\mathrm{d}h_{1FM}/\mathrm{d}x$, $\mathrm{d}h_{1F_0}/\mathrm{d}x$, and $\mathrm{d}h_{1b}/\mathrm{d}x$ are given by equations (42)–(44) and details of the calculations are given in appendix E. Here we give the asymptotic forms for the corresponding functions:

$$\lim_{x/w\gg 1}\frac{\mathrm{d}h_{1G}}{\mathrm{d}x} = -\frac{x}{a^2} + h_{1G}{}' + O\left(\frac{1}{x}\right), \tag{69}$$

$$\lim_{x/w\gg 1}\frac{\mathrm{d}h_{1M}}{\mathrm{d}x} \to \frac{2}{3}\frac{x}{a^2} + h_{1M}{}' + O\left(\frac{1}{x}\right), \tag{70}$$

$$\lim_{x/w\gg 1}\frac{\mathrm{d}h_{1FM}}{\mathrm{d}x} \to -\frac{4}{3}\frac{x}{a^2} + h_{1FM}{}' + O\left(\frac{1}{x}\right), \tag{71}$$

$$\lim_{x/w\gg 1}\frac{\mathrm{d}h_{1b}}{\mathrm{d}x} \to h_{1b}' + O\left(\frac{1}{x}\right). \tag{72}$$

$$\lim_{x/w\gg 1}\frac{\mathrm{d}h_{1F_0}}{\mathrm{d}x} \to \frac{x}{a^2}\left(\frac{1}{6} + \frac{1}{2}\ln\left(2\frac{x^2}{w^2}\right) + s\right) + O\left(\frac{1}{x}\right). \tag{73}$$

Here, the h_{1G}', h_{1M}', h_{1FM}', and h_{1b}' parameters are related to the numerical coefficients c_1, δ_{1M}, δ_{1G}, δ_{1FM} that are defined in the appendix E:

$$h_{1G}{}' = -\frac{w}{\pi\sqrt{2}\,a^2}sc_1 + \frac{w}{\sqrt{2}\,a^2}\delta_{1G}, \tag{74}$$

$$h_{1M}{}' = -\frac{w}{\sqrt{2}\,a^2}\delta_{1M}, \tag{75}$$

$$h_{1FM}{}' = \frac{w}{\sqrt{2}\,a^2}\delta_{1FM}. \tag{76}$$

$$h_{1b}{}' = \frac{w}{\pi\sqrt{2}\,b^2}c_1. \tag{77}$$

Collecting all terms, one has for the symmetric (even in x) matching function h_1:

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} = -\frac{3}{2}\frac{x}{a^2} + h_1{}', \tag{78}$$

where $h_1' = h_{1a}' + h_{1b}'$ and $h_{1a}' = h_{1G}' + h_{1M}' + h_{1FM}$. Note that $\mathrm{d}h_{1F0}/\mathrm{d}x$ part does not contribute to the nonlinear matching condition, but the first term in (73) is important to ensure the correct matching of the linear term $\sim x/a^2$ between nonlinear and linear solutions. From equations (B.5) and (68), one can see that the second terms in (73) is also well matched to the linear region. The third term in (73) is not considered here in detail, this is matched in the higher order. Note that our theory is formally based on small parameters, $\Delta' a < 1$, $\Sigma' a < 1$, so that $s < 1$.

The first term in the expression (78) is matched exactly to the fourth term in the outer solution (68), which confirms that the nonlinear solution can be matched to the outer region. Comparison of (78) and (68) gives the nonlinear matching condition $h_1' = \Delta'/2$, leading to the equation for the magnetic island width

$$\Delta' = \frac{\sqrt{2}\,c_1 w}{\pi a^2}\left[\pi\frac{g_1}{c_1} - s + \frac{a^2}{b^2}\right]. \tag{79}$$

where numerical coefficients $g_1 = \delta_{1G} - \delta_{1M} + \delta_{1FM} \simeq 1.84$ and $c_1 = 1.828$ giving $\pi g_1 / c_1 \simeq 3.16$. The closed form integral expressions for coefficients δ_{1G}, δ_{1M}, δ_{1FM} and c_1 are given by equations (E.4), (E.9), (E.14) and (E.5), respectively.

Additional term corresponding to the non-uniform resistivity (profile B) was calculated in the appendix G giving the equation

$$\Delta' = \frac{\sqrt{2}\,c_1 w}{\pi a^2}\left[\pi\frac{g_1}{c_1} - s + \frac{a^2}{b^2} + \lambda\frac{2g_{1B}}{c_1}\right]. \tag{80}$$

Similar to [10], we have introduced the λ parameter: $\lambda = 1$ for profile B and $\lambda = 0$ for profile A.

Using numerical values and s from equation (65) one obtains

$$\Delta' = 0.82\frac{w}{a^2}\left[3.62 - \ln\left(\frac{w}{a}\right) - \frac{a\Sigma'}{2} + \frac{a^2}{b^2} + 0.35\lambda\right]. \tag{81}$$

This equation has the same structure as [7, 10, 11] with one slighlty different numerical coefficient. The paper [10] gives the coefficient $4.42 - \ln(2) = 3.73$ instead of our 3.62.

7. Summary

Accurate nonlinear equation for magnetic island in Rutherford regime is important for many practical problems dealing with magnetic island excitation and control as well as for guiding and interpretations of numerical experiments on magnetic islands. Such equation has been derived earlier in a number of papers [2–11]. We revisited the problem of saturation of the magnetic island in configuration with a finite current gradient at the rational surface. Though our approach is within the general realm of the asymptotic theory, the actual approach is quite different from [3, 4, 6–11]. [7, 10, 11] do not give the expressions for numerical coefficients nor any details on how they were calculated. We have obtained the explicit form for the perturbed magnetic flux function describing the magnetic island that opens the way for the extension of the theory into the higher orders along the lines suggested in [21] as well as application to the problems of NTM control [19, 20]. Our analysis confirms the general structure of the nonlinear dispersion equation as obtained earlier in [7, 9–11]. Numerical values of our coefficient given by our analytical expressions is very close to that given in [10]. Further details of the comparison with [7, 10, 11] are given in appendix H. The analytical theory predicts the island width that is in fairly good agreement with results of numerical simulations that are discussed in Part II [23].

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Appendix A. Summary of auxiliary integrals and definitions

For convenience, we summarize here various definitions and expressions for axiliary functions and integrals

$$u = \frac{\psi}{\psi_1}, \tag{A.1}$$

$$\psi_0{}''x_0\mathrm{d}x_0 = \mathrm{d}\psi = \psi_1\mathrm{d}u, \tag{A.2}$$

$$x_0\mathrm{d}x_0 = \frac{w^2}{4}\,\mathrm{sgn}\,(x)\,\mathrm{d}u, \tag{A.3}$$

$$x_0 = \frac{w}{\sqrt{2}}\,\mathrm{sgn}\,(x)\sqrt{u+\cos\xi}, \tag{A.4}$$

$$\mathrm{d}x_0 = \frac{w}{2\sqrt{2}}\,\mathrm{sgn}\,(x)\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}}, \tag{A.5}$$

$$\langle x_0^{-1}\rangle = \frac{\sqrt{2}}{w}\int_0^{\xi_m(u)}\frac{\mathrm{d}\xi}{\sqrt{u+\cos\xi}} = \frac{\sqrt{2}}{w}\beta(u), \tag{A.6}$$

$$\langle x_0^{-1}\cos\xi\rangle = \frac{\sqrt{2}}{w}\int_0^{\xi_m(u)}\frac{\cos\xi\mathrm{d}\xi}{\sqrt{u+\cos\xi}} = \frac{\sqrt{2}}{w}\alpha(u), \tag{A.7}$$

$$\frac{\langle 1\rangle}{\langle x_0^{-1}\rangle} = \frac{w\pi}{\sqrt{2}}\frac{\sigma(u-1)}{\beta(u)}, \tag{A.8}$$

$$\frac{\langle x_0^{-1}\cos\xi\rangle}{\langle x_0^{-1}\rangle} = \frac{\alpha(u)}{\beta(u)}, \tag{A.9}$$

$$\alpha(u) = \int_0^{\xi_m(u)}\cos\xi\frac{\mathrm{d}\xi}{\sqrt{u+\cos\xi}}, \tag{A.10}$$

$$\beta(u) = \int_0^{\xi_m(u)}\frac{\mathrm{d}\xi}{\sqrt{u+\cos\xi}}, \tag{A.11}$$

$$\beta_3(u) = \int_0^{\xi_m(u)}\frac{\mathrm{d}\xi}{(u+\cos\xi)^{3/2}}, \tag{A.12}$$

$$\beta_{-1}(u) = \int_0^{\xi_m(u)}\sqrt{u+\cos\xi}\,\mathrm{d}\xi, \tag{A.13}$$

$$\alpha_{-1}(u) = \int_0^{\xi_m(u)}\cos\xi\sqrt{u+\cos\xi}\,\mathrm{d}\xi, \tag{A.14}$$

$$\alpha_3(u) = \int_0^{\xi_m}\frac{\cos\xi}{(u+\cos\xi)^{3/2}}\mathrm{d}\xi, \tag{A.15}$$

$$\varepsilon(u',u) = \int_0^{\pi}\frac{\mathrm{d}\xi}{\sqrt{u'+\cos\xi}\sqrt{u+\cos\xi}}. \tag{A.16}$$

and

$$\xi_m = \pi,\ \text{for}\ u > 1, \tag{A.17}$$

$$\xi_m = \cos^{-1}(u),\ \text{for}\ u < 1. \tag{A.18}$$

Function $\xi_{m'}$ is defined by the expressions

$$\xi_{m'} = \arccos\left(2\frac{x^2}{w^2} - u\right)\ \text{for}\ 2\frac{x^2}{w^2} - 1 < u < 2\frac{x^2}{w^2} + 1, \tag{A.19}$$

and

$$\xi_{m'} = 0 \text{ for } u < 2\frac{x^2}{w^2} - 1. \quad \text{(A.20)}$$

Appendix B. Linear solution in the outer region

Here we consider the general structure of the outer solution taking into account a finite current gradient at the rational surface. The linear equation in the outer region can be obtained by linearizing the MHD equilibrium equation (1), resulting in

$$(\widetilde{\psi}'' - k^2\widetilde{\psi}) - \frac{\partial^3\psi_0(x)/\partial x^3}{\partial\psi_0(x)/\partial x}\widetilde{\psi} = 0, \quad \text{(B.1)}$$

where k is the poloidal wave-vector and $\psi = \psi_0(x)$ is the equilibrium flux function. Using the equilibrium in equation (13) one gets

$$(\widetilde{\psi}'' - k^2\widetilde{\psi}) - \frac{\psi_0'''(0)}{x\psi_0'(0) + x^2\psi_0'''(0)/2}\widetilde{\psi} = 0. \quad \text{(B.2)}$$

The term with $\psi_0^{\mathrm{IV}}x^4$ has been neglected here. Expanding for small x we obtain the linear equation in the outer region

$$\widetilde{\psi}'' - \left(\frac{1}{ax} - \frac{1}{2a^2} + k^2\right)\widetilde{\psi} = 0, \quad \text{(B.3)}$$

where $a^{-1} = J'_0/J_0$.

The series solution of (B.3) has the form

$$\begin{aligned}\psi(x) &= 1 + x^2\left(-\frac{1}{a^2} + \frac{k^2}{2}\right) + O(x^3)\\ &+ \frac{x}{a}\ln\left|\frac{x}{a}\right|\left(1 + \frac{x}{2a} + \frac{1}{6}k^2x^2 + O(x^3)\right)\\ &+ C_\pm x\left(1 + \frac{x}{2a} + \frac{1}{6}k^2x^2 + O(x^3)\right).\end{aligned} \quad \text{(B.4)}$$

Here, the solution was normalized such that $\psi(x = 0) = 1$. Note the normalization in the logarithmic term can be changed by redefining the $C_\pm$ coefficients. More exactly, this corresponds to redefining the anti-symmetric (odd) part of the solution characterized by Σ' parameter. As has been noted in [7, 10, 11], the Σ' parameter depends on the choice of the length L in equation (66); which in principle can be arbitrary. Here, we will be using the fixed length a in the logarithm term, where a is the current gradient length scale in the outer region.

From equation (B.4), one can calculate the following expressions for the derivatives of the flux function

$$\begin{aligned}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} &= C_\pm\left(1 + \frac{x}{a} + \frac{1}{2}k^2x^2\right)\\ &+ \frac{1}{a}\ln\left|\frac{x}{a}\right|\left(1 + \frac{x}{a}\right) + \frac{1}{a} - \frac{3}{2}\frac{x}{a^2},\end{aligned} \quad \text{(B.5)}$$

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = C_\pm\left(\frac{1}{a} + xk^2\right) + \frac{1}{ax} - \frac{1}{2a^2}. \quad \text{(B.6)}$$

Coefficients $C_\pm$ are determined by global boundary conditions, for $x > 0$ and $x < 0$, respectively. They are related to the Δ' and Σ' parameters, $C_\pm = (\Sigma' \pm \Delta')/2$. The Δ' is a standard tearing mode stability parameter

$$\Delta' = \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{x\to 0+} - \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{x\to 0-}. \quad \text{(B.7)}$$

The definition of Σ' depends on the normalization in the logarithm argument: when the form $\ln(x/a)$ is used, the Σ' matching parameter is given by

$$\Sigma' = \lim_{\varepsilon\to 0}\left(\left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{\varepsilon} + \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{-\varepsilon} - \frac{2}{a}\ln\left|\frac{\varepsilon}{a}\right| - \frac{2}{a}\right). \quad \text{(B.8)}$$

Thus, in the leading order, the outer solution for $x \to 0$ can be written

$$\psi^\pm(x) = 1 + \frac{\Sigma'}{2}x + \frac{x}{a}\ln\left|\frac{x}{a}\right| \pm \frac{\Delta'}{2}x, \quad \text{(B.9)}$$

for $x > 0$ and $x < 0$, respectively. The Δ' describes the symmetric (even) part of the asymptotic, while the Σ' is the anti-symmetric (odd) part. The logarithmic term $x\ln(x/a)/a$ is also anti-symmetric. The explicit form of the next order expression for the flux derivative is given by equation (63).

Appendix C. Nonlinear current and asymptotic in the inner region

The nonlinear equation in the inner region has the form

$$\frac{\partial^2}{\partial x^2}H(x,\xi) = F(x,\xi) + (k^2 - b^{-2})\cos\xi, \quad \text{(C.1)}$$

Where

$$F(x,\xi) = F_0(u) + F_{1G}(u) + F_{1M}(u) + F_{1b}(u), \quad \text{(C.2)}$$

and $F_0(u)$, $F_{1G}(u)$, $F_{1M}(u)$ and $F_{1b}(u)$ are nonlinear current functions defined in equations (26)–(29). The explicit expressions for these functions are written by using the lowest order expressions for functions M and G and the definitions of the flux functions in appendix A.

From (26) one finds

$$F_0(x,\xi) = \frac{4}{w^2a}\left(x - \mathrm{sgn}\,(x)\frac{\pi w}{\sqrt{2}\beta(u)}\sigma(u-1)\right), \quad \text{(C.3)}$$

where $u = u(x, \xi)$.

Using the expression (30) for G function in (), the flux average of G is written as

$$\langle G\rangle = -\frac{\sqrt{2}\pi w}{4a}\int_1^u\frac{\varepsilon(u,u')\,\mathrm{d}u'}{\beta(u')} + \frac{\sqrt{2}w}{4a}s\alpha(u) \quad \text{(C.4)}$$

Respectively, the function F_{1G} takes the form

$$F_{1G} = \frac{1}{a^2}\left[s\frac{\alpha(u)}{\beta(u)} - \frac{\pi}{\beta(u)}\int_1^u\frac{\varepsilon(u,u')\,\mathrm{d}u'}{\beta(u')}\right]. \quad \text{(C.5)}$$

From equation (53), the weighted average of the function M is

$$\langle x_0^{-1}M\rangle = \frac{\pi}{2a}\beta_3(u)\int_1^u\frac{\mathrm{d}u}{\beta(u)}, \quad \text{(C.6)}$$

so that the equation (28) gives for $F_{1M}(u)$

$$F_{1M}(u) = -\frac{\pi^2}{a^2}\frac{\beta_3(u)}{\beta^2(u)}\int_1^u \frac{du'}{\beta(u')}. \qquad (C.7)$$

Expression for $F_{1b}(u)$ is obtained by using the flux functions defined in appendix A.

$$F_{1b}(u) = b^{-2}\frac{\langle x_0^{-1}\cos\xi\rangle}{\langle x_0^{-1}\rangle} = \frac{\alpha(u)}{b^2\beta(u)}. \qquad (C.8)$$

It is instructive to confirm that nonlinear equation (C.1) reduces to the linear equation (64) in the outer region. This is done by considering asymptotic limits of the functions $F_0(u)$, $F_{1G}(u)$, $F_{1M}(u)$ and $F_{1b}(u)$ in the limit $x \gg w$.

Expanding the function $\beta(u)$ for large u, one obtains

$$\lim_{u\gg1} F_0(u) = -\frac{2\sqrt{2}}{wa}\sqrt{u} = -\frac{4x}{w^2a}\left(\frac{1}{6}\frac{x}{a} - \frac{1}{4x^2/w^2}\cos\xi + \frac{1}{24}\frac{w^2}{ax}\cos\xi - \frac{1}{4x^2/w^2}H(x,\xi)\right) \qquad (C.9)$$

We have used here the auxiliary expansion for $\sqrt{u}$ in the limit $x \gg w$,

$$\lim_{u\gg1}\sqrt{u} = \sqrt{2}\frac{x}{w}\left(1 + \frac{1}{6}\frac{x}{a} - \frac{1}{4x^2/w^2}\cos\xi + \frac{1}{24}\frac{w^2}{ax}\cos\xi - \frac{1}{4x^2/w^2}H(x,\xi)\right) \qquad (C.10)$$

The large u expansions for $\langle G\rangle$ and $\langle x_0^{-1}M\rangle$ have the form

$$\lim_{u\gg1}\langle G\rangle = -\frac{w\pi}{2\sqrt{2}a}\sqrt{u}, \qquad (C.11)$$

$$\lim_{u\gg1}\langle x_0^{-1}M\rangle = \frac{\pi}{3a}. \qquad (C.12)$$

Similarly, nonlinear functions $F_{1G}(u)$ and $F_{1M}(u)$ are

$$\lim_{u\gg1} F_{1G}(u) = \frac{u}{a^2} = \frac{1}{a^2}\left(2\frac{x^2}{w^2} - \cos\xi\right) + O\left(\frac{1}{x^2}\right), \qquad (C.13)$$

$$\lim_{u\gg1} F_{1M}(u) = -\frac{2}{3}\frac{u}{a^2} = -\frac{2}{3a^2}\left(2\frac{x^2}{w^2} - \cos\xi\right) + O\left(\frac{1}{x^2}\right). \qquad (C.14)$$

The function $F_{1b}(u)$ is strictly localized in the nonlinear region and has no linear part:

$$\lim_{u\gg1} F_{1b}(u) = O\left(\frac{1}{x^2}\right). \qquad (C.15)$$

Collecting all terms in (C.9), (C.13) and (C.14) we find that the nonlinear equation (C.1) in the linear limit $x \gg w$ reduces to the form

$$\frac{\partial^2}{\partial x^2}H(x,\xi) = \frac{\cos\xi}{ax} - \frac{\cos\xi}{2a^2} + (k^2 - b^{-2})\cos\xi, \qquad (C.16)$$

which is identical to the linear equation (B.3) in the outer region.

Appendix D. The asymptotics of the zeroth order solution in nonlinear region

The zeroth order solution, the function $H_0^{nl}(x,\xi)$, is given by the equation (52). This is a form used to calculate the next order functions in equations (42)–(46). The asymptotic form of $H_0^{nl}(x,\xi)$ is not required for our purposes, but we will give it here to provide more information about its structure.

The following expansions for $x \gg w$ will be useful:

$$\int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{du'}{\beta(u')} = \frac{2}{3}\frac{u^{3/2}}{\pi}\left(1 + O\left(\frac{1}{u^2}\right)\right); \qquad (D.1)$$

and

$$\int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{du'}{\beta(u')\sqrt{u'+\cos\xi}} = \int_{-\cos\xi}^{2x^2/w^2-\cos\xi}\frac{\sqrt{u'}\,[1+O(1/u'2)]\,du'}{\pi\sqrt{u'}}\left(1 - \frac{\cos\xi}{2u'}\right) = \frac{u}{\pi} - \frac{\cos\xi}{2\pi}(\ln(u) + G_0), \qquad (D.2)$$

where G_0 is a constant analogous to g_0, see equation (D.12). Then the function $H_0^{nl}(x,\xi)$ in (52) takes the form

$$H_0^{nl}(x\xi) = \frac{2x^3}{3w^2a} - \frac{x}{a}\left(2\frac{x^2}{w^2} - \cos\xi\right) + \frac{x}{2a}\ln(u)\cos\xi + \frac{2w}{3\sqrt{2}a}\left(2\frac{x^2}{w^2} - \cos\xi\right)^{3/2} + s\frac{x}{a}\cos\xi. \qquad (D.3)$$

Collecting all terms and expanding for $x \gg w$ one obtains

$$H_0^{nl}(x,\xi) = s\frac{x}{a}\cos\xi + \frac{x}{2a}\left(\ln\left(2\frac{x^2}{w^2}\right) + G_0\right)\cos\xi. \qquad (D.4)$$

For matching purposes we require the first harmonics of the lowest order

$$h^a(x) = \frac{2}{\pi}\int_0^\pi \cos\xi d\xi H_0(x,\xi). \qquad (D.5)$$

Using (50), one has

$$h^a(x) = -\frac{2x}{a}\int_1^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi} d\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}} + \frac{\sqrt{2}w}{a}\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi} d\xi\cos\xi + s\frac{x}{a}. \qquad (D.6)$$

In what follows we describe the derivation of the asymptotics of $h^a(x)$ which is required to determine the s coefficient in the inner solution. The first term in equation (D.6) can be written as a sum of two integrals

$$\int_1^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi}d\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}} = \int_1^{2x^2/w^2-1}\frac{du'}{\beta(u')}\int_0^{\pi}d\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}} + \int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi}d\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}}. \quad (D.7)$$

Then, equation (D.6) becomes

$$h^a = -\frac{2x}{a}\int_1^{2x^2/w^2-1}\frac{\alpha(u')\,du'}{\beta(u')} - \frac{2x}{a}\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi}d\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}} + \frac{\sqrt{2}w}{a}\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi}d\xi\cos\xi + s\frac{x}{a}. \quad (D.8)$$

Asymptotic of $\beta(u)$ for large x (and large u) is $\beta(u) = \pi/\sqrt{u}$. This gives

$$\beta^{-1} \simeq \pi^{-1}\left(2\frac{x^2}{w^2} - \cos\xi\right)^{1/2} \simeq \frac{\sqrt{2}x}{\pi w}\left(1 - \frac{w^2\cos\xi}{4x^2}\right). \quad (D.9)$$

For large u, and taking into account (D.9), the second term in (D.8) is transformed as follows

$$\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du'}{\beta(u')}\int_{\xi_{m'}}^{\pi}d\xi\frac{\cos\xi}{\sqrt{u'+\cos\xi}} = \int_{2x^2/w^2-1}^{2x^2/w^2+1}du\frac{1}{\beta(u)\sqrt{u}}\int_{\xi_{m'}}^{\pi}d\xi\cos\xi = -\frac{1}{\pi}\int_{2x^2/w^2-1}^{2x^2/w^2+1}\sin\xi_{m'}\,du. \quad (D.10)$$

As a result, this term cancels with the third term (D.8).

With $\alpha(u) = -\pi/(4u^{3/2})$ and (D.9), one has for large $u \simeq 2x^2/w^2$

$$\frac{\alpha}{\beta} = -\frac{1}{4u}, \quad (D.11)$$

and

$$\lim_{x/w\gg 1}\int_1^{2x^2/w^2-1}\frac{du'}{\beta(u')}\alpha(u') = -\frac{1}{4}\ln u + g_0 = -\frac{1}{4}\ln\left(2\frac{x^2}{w^2} - 1\right) + g_0 \simeq -\frac{1}{4}\ln\left(2\frac{x^2}{w^2}\right) + g_0, \quad (D.12)$$

where $g_0 = -0.06$ is constant calculated numerically.

Finally, the asymptotics of $h^a(x)$, the antisymmetric part of the first harmonic of the nonlinear solution, becomes

$$h_0 = \lim_{x/w\gg 1} h^a(x) = \frac{x}{2a}\ln\left(\frac{2x^2}{w^2}\right) - \frac{2x}{a}g_0 = \frac{x}{a}\ln\left|\frac{x}{w}\right| + \frac{x}{a}\left(\frac{\ln 2}{2} - 2g_0\right) + s\frac{x}{a}. \quad (D.13)$$

Appendix E. First order matching functions and asymptotic

Here, we give some details of calculations of nonlinear matching functions in equations (43)–(46). Changing the order of integration as in (58), equation (42) gives for dh_{1G}/dx

$$\frac{dh_{1G}}{dx} = \frac{w}{\sqrt{2}\pi}\int_0^{\pi}\cos\xi\,d\xi\int_{-\cos\xi}^{u_0(x,\xi)}F_{1G}(u)\frac{du}{\sqrt{u+\cos\xi}} = \frac{w}{\sqrt{2}\pi}\left[\int_{-1}^{2x^2/w^2-1}F_{1G}(u)\,\alpha(u)\,du + \int_{2x^2/w^2-1}^{2x^2/w^2+1}F_{1G}(u)\int_{\xi_{m'}}^{\pi}\frac{\cos\xi\,d\xi}{\sqrt{u+\cos\xi}}\right]. \quad (E.1)$$

Using the explicit expression (C.5) for F_{1G}, the first term on RHS of (E.1) becomes

$$\int_{-1}^{2x^2/w^2-1}F_{1G}(u)\,\alpha(u)\,du = \frac{1}{a^2}\left[-s\int_{-1}^{2x^2/w^2-1}du\frac{\alpha^2(u)}{\beta(u)} + \int_1^{2x^2/w^2-1}du\frac{\pi\alpha(u)}{\beta(u)}\int_1^u\frac{\varepsilon(u',u)du'}{\beta(u')}\right]. \quad (E.2)$$

For large $x \gg w$, this integral has an asymptotic limit

$$\int_{-1}^{2x^2/w^2-1}F_{1G}(u)\,\alpha(u)\,du = \frac{1}{a^2}\left(-c_1 s - \frac{\sqrt{2}\pi}{2}\frac{x}{w} + \pi\delta_{1G}\right) + O\left(\frac{w}{x}\right), \quad (E.3)$$

where

$$\delta_{1G} = \lim_{t\to\infty}\left[\int_1^t du\frac{\alpha(u)}{\beta(u)}\int_1^u\frac{\varepsilon(u,u')}{\beta(u')}du' + \frac{1}{2}\sqrt{t}\right] \simeq 0.93, \quad (E.4)$$

and

$$c_1 = \lim_{t\to\infty}\left[\int_{-1}^t du\frac{\alpha^2(u)}{\beta(u)}\right]. \quad (E.5)$$

The second term in (E.1) in the same limit is evaluated using (C.13). It gives

$$\int_{2x^2/w^2-1}^{2x^2/w^2+1}F_{1G}(u)\int_{\xi_{m'}}^{\pi}\frac{\cos\xi\,d\xi}{\sqrt{u+\cos\xi}} = -\pi\frac{\sqrt{2}}{2}\frac{x}{a^2 w} + O\left(\frac{w}{x}\right). \quad (E.6)$$

Collecting terms in (E.1)-(E.6) one obtains the expression (69).

The expression for dh_{1M}/dx is evaluated in a similar way:

$$\frac{dh_{1M}}{dx}=\frac{w}{\sqrt{2}\pi}\int_0^{\pi}\cos\xi\,d\xi\int_{-\cos\xi}^{u_0(x,\xi)}F_{1M}(u)\frac{du}{\sqrt{u+\cos\xi}}$$
$$=\frac{w}{\sqrt{2}\pi}\left[\int_{-1}^{2x^2/w^2-1}F_{1M}(u)\,\alpha(u)\,du+\int_{2x^2/w^2-1}^{2x^2/w^2+1}F_{1M}(u)\int_{\xi_{m'}}^{\pi}\frac{\cos\xi\,d\xi}{\sqrt{u+\cos\xi}}\right]. \tag{E.7}$$

The first term here becomes

$$\int_1^{2x^2/w^2-1}F_{1M}(u)\,\alpha(u)\,du=\frac{\pi\sqrt{2}}{3a^2}\frac{x}{w}-\delta_{1M}\frac{\pi}{a^2}+O\left(\frac{w}{x}\right), \tag{E.8}$$

where

$$\delta_{1M}=\lim_{t\to\infty}\left[\pi\int_1^t d\,u\frac{\alpha(u)\beta_3(u)}{\beta^2(u)}\int_1^u\frac{du'}{\beta(u')}+\frac{1}{3}\sqrt{t}\right]\simeq 0.46. \tag{E.9}$$

From (C.14) it follows that

$$\int_{2x^2/w^2-1}^{2x^2/w^2+1}F_{1M}(u)\int_{\xi_{m'}}^{\pi}\frac{\cos\xi\,d\xi}{\sqrt{u+\cos\xi}}=\frac{\pi\sqrt{2}}{3a^2}\frac{x}{w}+O\left(\frac{w}{x}\right). \tag{E.10}$$

Collecting (E.8) and (E.10), one obtains the expression (70).

The first order matching function h_{1FM} is a result of the transformation of variables (21) in equation (39):

$$\frac{dh_{1FM}}{dx}=-\frac{w}{\sqrt{2}\pi}\int_0^{\pi}\cos\xi\,d\xi\int_{-\cos\xi}^{u_0(x,\xi)}F_0(u)M(u,\xi)\frac{du}{\sqrt{u+\cos\xi}} \tag{E.11}$$

Using the definitions (26) and (53) for $F_0(u)$ and $M(u,\xi)$, and reversing the order of integration according to (58), the expression (E.11) takes the form

$$\frac{dh_{1FM}}{dx}=\frac{\pi w}{\sqrt{2}a^2}\left[\int_1^{2x^2/w^2-1}d\,u\frac{\alpha_3(u)}{\beta(u)}\int_1^u\frac{du'}{\beta(u')}+\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du}{\beta(u)}\int_{\xi_{m'}}^{\pi}\frac{\cos\xi\,d\xi}{(u+\cos\xi)^{3/2}}\int_1^u\frac{du'}{\beta(u')}\right] \tag{E.12}$$

The first term on the right hand side for large $x\gg w$ is presented in the form

$$\int_1^{2x^2/w^2-1}d\,u\frac{\alpha_3(u)}{\beta(u)}\int_1^u\frac{du'}{\beta(u')}=-\frac{1}{\pi}\sqrt{2}\frac{x}{w}+\frac{\delta_{1FM}}{\pi}+O\left(\frac{w}{x}\right), \tag{E.13}$$

where the numerical coefficient δ_{1FM} is defined by the following integral

$$\delta_{1FM}=\lim_{t\to\infty}\left[\pi\int_1^t\frac{\alpha_3(u)}{\beta(u)}du\int_1^u\frac{du'}{\beta(u')}+\sqrt{t}\right]=1.42. \tag{E.14}$$

The second term in (E.12) for large $x\gg w$ is evaluated to

$$\int_{2x^2/w^2-1}^{2x^2/w^2+1}\frac{du}{\beta(u)}\int_{\xi_{m'}}^{\pi}\frac{\cos\xi d\xi}{(u+\cos\xi)^{3/2}}\int_1^u\frac{du'}{\beta(u')}=-\frac{1}{3\pi}\sqrt{2}\frac{x}{w}+O\left(\frac{w}{x}\right). \tag{E.15}$$

Finally, from (E.12), (E.13) and (E.15) one obtains the expression (71).

The matching function dh_{1F_0}/dx given by equation (46) is evaluated in the limit $x\gg w$ by the expansion of the upper limit of the inner integral, $u_0(x,\xi)\gg u_1(x,\xi)$, so that

$$\frac{dh_{1F_0}}{dx}=\frac{w}{\sqrt{2}\pi}\int_0^{\pi}\cos\xi\,d\xi\int_{u_0(x,\xi)}^{u_0(x,\xi)+u_1(x,\xi)}F_0(u)\frac{du}{\sqrt{u+\cos\xi}}$$
$$\simeq\frac{w}{\sqrt{2}\pi}\int_0^{\pi}\cos\xi\,d\xi\left.\frac{F_0(u)}{\sqrt{u+\cos\xi}}\right|_{u=u_0(x,\xi)}u_1(x,\xi). \tag{E.16}$$

From definitions in (18) and (50), one finds in the limit $x\gg w$ the expression for $u_1(x,\xi)$

$$u_1(x,\xi)=\frac{x}{a}\left(u-\frac{1}{2}\cos\xi\ln u\right)-\frac{\sqrt{2}w}{3a}u^{3/2}-s\frac{x}{a}\cos\xi. \tag{E.17}$$

In the required ordering

$$F_0(u)=-\frac{2\sqrt{2}}{wa}\sqrt{u}. \tag{E.18}$$

Using (E.17) and (E.18) in (E.16) one finds the expression (73). We have used in (E.16) the following equations valid for $u\gg 1$

$$\int_0^{\pi}u^{3/2}\cos\xi\,d\xi=-\frac{3\sqrt{2}\pi}{4}\frac{x}{w}, \tag{E.19}$$

$$\int_0^{\pi}u^2\cos\xi\,d\xi=-\frac{2\pi x^2}{w^2}. \tag{E.20}$$

Appendix F. Alternative calculation of dh_{1F0}/dx and dh_{1FM} in equation (40)

The first order terms dh_{1F_0}/dx and dh_{1FM} that are produced from the integral of $F_0(u)$ in equation (39) can also be obtained by the alternative method. The argument of the function $F_0(u)$ involves the higher order corrections which mix the parity of the function $u=u_0(x,\xi)+u_1(x,\xi)$, and hence the parity of $F_0(u)$. To separate the parities, it is convenient to expand F_0 with respect to $u_1 \ll u_0$ and calculate the intergral $\int_0^x F_0(u,\xi)\,\mathrm{d}x$ via the transformation to the symmetric variable $u = u_0(x, \xi)$ as was commented in [11]. Expanding $F_0(u_0(x, \xi) + u_1(x, \xi), \xi)$ and integrating by parts one obtains

$$\begin{aligned}\int_0^x F_0(u,\xi)\,\mathrm{d}x &= \int_{-\cos\xi}^{u_0(x,\xi)} F_0(u)\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}} \\ &+ \int_{-\cos\xi}^{u_0(x,\xi)} \frac{\partial}{\partial u}F_0\bigg|_{u=u_0(x,\xi)} \frac{u_1(x_0(u,\xi),\xi)}{\sqrt{u+\cos\xi}}\mathrm{d}u \\ &= \int_{-\cos\xi}^{u_0(x,\xi)} F_0(u)\frac{\mathrm{d}u}{\sqrt{u+\cos\xi}} \\ &+ F_0\bigg|_{u_0(x,\xi)} \frac{u_1(x_0(u,\xi),\xi)}{\sqrt{u+\cos\xi}}\bigg|_{-\cos\xi}^{u_0(x,\xi)} \\ &- \int_{-\cos\xi}^{u_0(x,\xi)} F_0\bigg|_{u=u_0(x,\xi)} \frac{\partial}{\partial u}\left(\frac{u_1(x_0(u,\xi),\xi)}{\sqrt{u+\cos\xi}}\right)\mathrm{d}u\end{aligned} \tag{F.1}$$

The first term here is equation (41); the second term corresponds to the $\mathrm{d}h_{1F0}/\mathrm{d}x$ term, equation (46); and the last term corresponds to the $\mathrm{d}h_{1FM}/\mathrm{d}x$ term, equation (45), which was derived in Section IV via different variable transformation.

Appendix G. Non-uniform resistivity model

In model B, the full current is determined from the equation $E_0 = \eta(x)\, J(x)$ giving

$$J(\psi) = E_0\frac{\langle(\partial\psi/\partial x)^{-1}\rangle}{\langle\eta(x)(\partial\psi/\partial x)^{-1}\rangle} = \frac{\langle(\partial\psi/\partial x)^{-1}\rangle}{\langle J_0^{-1}(x)(\partial\psi/\partial x)^{-1}\rangle}. \tag{G.1}$$

Using the expansions

$$J(x) = J_0 + J_0{}'x + J_0{}''\frac{x^2}{2}, \tag{G.2}$$

and

$$\eta(x) = \eta_0 + \eta_0{}'x + \eta_0{}''\frac{x^2}{2}, \tag{G.3}$$

one obtains [7, 11] for $J(\psi)$

$$\begin{aligned}J(\psi) = J_0\Bigg(1 &+ \frac{1}{a}\frac{\langle x(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle} + \frac{1}{a^2}\left[\left(\frac{\langle x(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\right)^2 \right.\\ &\left.- \frac{\langle x^2(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\right]\Bigg) + \frac{J_0}{2b^2}\frac{\langle x^2(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle},\end{aligned} \tag{G.4}$$

where $a = J_0{}'/J_0$ and $b^{-2} = -J_0''/J_0$. The required nonlinear equation then is

$$\begin{aligned}\frac{\partial^2}{\partial x^2}H(x,\xi) + \frac{\partial^2 H(x,\xi)}{\partial y^2} &= k^2\cos\xi - \frac{4}{w^2 a}\left(\frac{\langle x(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle} - x\right) \\ &- \frac{4}{w^2a^2}\left[\left(\frac{\langle x(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\right)^2 - \frac{\langle x^2(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\right] \\ &+ \frac{2}{w^2b^2}\left(\frac{\langle x^2(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle} - x^2\right).\end{aligned} \tag{G.5}$$

Thus, the difference between the profile A and profile B cases is reduced to the appearance in (G.4) of an additional term proportional to a^{-2}, the term in square brackets. Thus, the equation (24) for profile B takes the form:

$$\frac{\partial^2}{\partial x^2}H(x,\xi) = F(x,\xi), \tag{G.6}$$

$$F(x,\xi) = F_0(u) + F_{1a}(u) + F_{1b}(u) + F_B, \tag{G.7}$$

where $F_{1a} = F_{1G} + F_{1M}$ and the additional term F_B is given by

$$F_B = -\frac{4}{w^2a^2}\left[\left(\frac{\langle x(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\right)^2 - \frac{\langle x^2(\partial\psi/\partial x)^{-1}\rangle}{\langle(\partial\psi/\partial x)^{-1}\rangle}\right]. \tag{G.8}$$

Corresponding matching parameter is

$$h_{1B}{}' = \frac{2}{\pi}\int_0^\pi \cos\xi \int_0^\infty F_B(u)\,\mathrm{d}x_0 = \frac{\sqrt{2}\,w}{\pi a^2}g_{1B}, \tag{G.9}$$

$$\begin{aligned}g_{1B} &= \int_{-1}^1 \mathrm{d}u\frac{\alpha(u)\beta_{-1}(u)}{\beta(u)} + \int_1^\infty \mathrm{d}u\left(\frac{\beta_{-1}(u)}{\beta(u)} - \frac{\pi^2}{\beta^2(u)}\right)\alpha(u) \\ &= 0.316\end{aligned} \tag{G.10}$$

Thus, for profile B, the contribution of $h_{1B}{}'$ is added to the matching condition in (38), producing an additional term in equation (81).

Appendix H. Additional comments on comparison with other works

For the benefit of readers, here we provide additional details to facilitate the comparison of our work with previous results.

As noted above our final equation is similar to the one in [7, 10, 11]. Note that these authors used w for the full width of the island, while in our paper, w is a half-width, so thus we have the factor 0.82 instead of 0.41 in [7, 10, 11].

The logarithmic term, in equation (81), $\ln(w/a)$, was obtained earlier in [3], see also [12–14]. We have obtained this term with numerical coefficient identical to [7, 10, 11], where it was written with arbitrary normalization factor, $\ln(w/r_0)$. The choice of the normalization factor affect the value of the Σ' parameter. It was emphasized in [7, 10, 11] that the combination $\ln(w/a) + a\Sigma'/2$ remains independent of the choice of r_0. In our work we use fixed normalization $\ln(w/a)$, where the a parameter is related to the current gradient, $a = J_0/J_0$. The term due to the asymmetry in the outer solution $\Sigma' a/2$ was present in the original work [2] with a different coefficient; in the current form, as in equation (81), it was obtained in [5].

The b^2/a^2 term in equation (81), related to the second derivative of the equilibrium current was included in the theory of [2]; in its present form, this term was derived in [6, 8]. Our numerical coefficient for the a^{-2} term in equation (81) is defined by closed form integral expression defined in appendix E. The numerical value of the coefficient is slightly different from those in [7, 10, 11]. Unfortunately, it is not possible to compare the details of the derivations between our work and [7, 10, 11]. The [7, 11] do not give any details of the derivations nor provide the full structure of the solution. The method of [10] is different from [7, 11]. References

[7, 10, 11] do not describe what is involved in calculations of numerical coefficients entering the final equation, e.g. equation (51) or equation (58) in [10], nor provide details of how the numerical values were obtained. We have obtained exactly the same numerical coefficient for the term responsible for the difference between profiles A and B as in [10], where it is written separately; only profile B is considered in [7, 11]. The numerical coefficients given in [7, 11 and 10] for profile B are slightly different: 4.85 in [7, 11] and 4.76 in [10].

We have demonstrated that our nonlinear equation (C.1) in the limit $w \gg a$ reduces to the linear equation (B.3) in the outer region (for details see appendix C), that confirms that the full solution of the nonlinear equation will reduce to the linear solution in the outer region. The matching of the derivatives of the flux functions, (68) and (78), involve the constant and diverging (linear in x) terms. In our approach, the diverging (linear in x) terms are generated from $\mathrm{d}h_{1G}/\mathrm{d}x$, $\mathrm{d}h_{1M}/\mathrm{d}x$, $\mathrm{d}h_{1FM}/\mathrm{d}x$ and $\mathrm{d}h_{1F0}/\mathrm{d}x$ terms in equations (42)–(46). Sum of these terms gives $-3x/(2a^2)$ which exactly matches to the corresponding term in the outer solution, compare equations (68) and (78). The importance of matching of such divergent terms has been commented upon in [7, 10, 11], however, it appears, that the treatment of these terms is different between [7, 11 and 10]. Only general discussion but no expressions is given in [7, 11]. Reference [10] has only asymptotic limit of the diverging terms and we unable to precisely compare our expressions with those in [10].

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